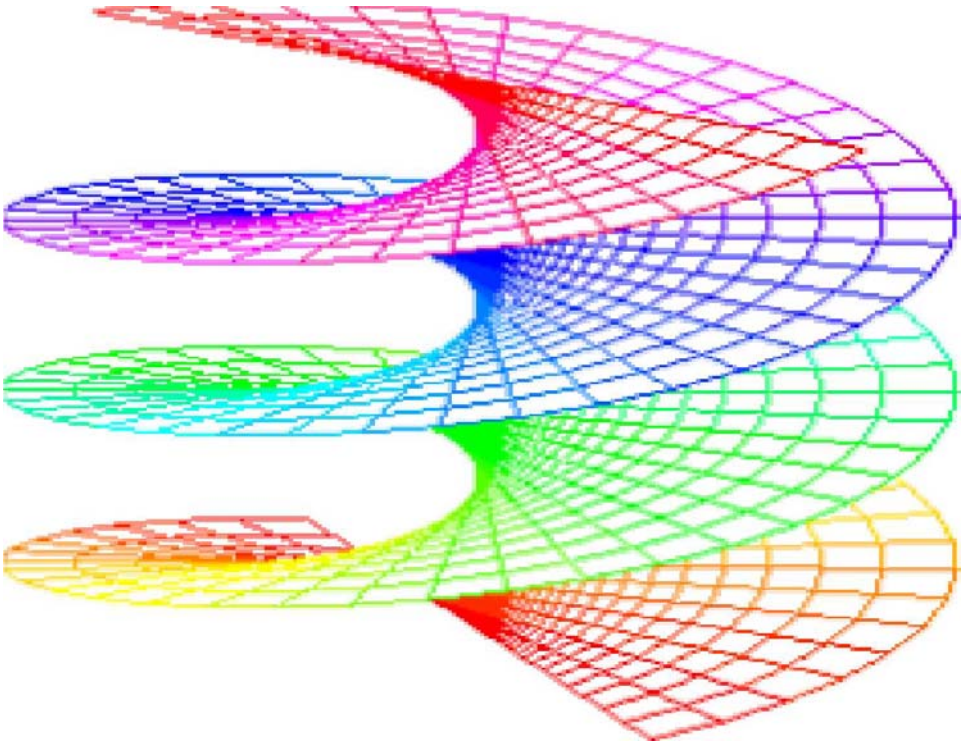
*ZHANG WENPENG*
*HAILONG LI*
**editors**

# *Scientia Magna*

## International Book Series

## Vol. 3, No. 1

**Third International Conference on Number Theory and Smarandache Problems**
**March 23-25, 2007**
**Weinan Teacher's University, P. R. China**



**2007**

**Editors:**

**Dr. Zhang Wenpeng**
**Department of Mathematics**
**Northwest University**
**Xi'an, Shaanxi, P. R. China**

**Dr. Hailong Li**
**Department of Mathematics**
**Weinan Theacher's University**
**P. R. China**

# Scientia Magna
## - international book series (Vol. 3, No. 1) -

**Third International Conference on Number Theory and Smarandache Problems**
**March 23-25, 2007**
**Weinan Teacher's University, P. R. China**

High American Press

**2007**

This book can be ordered in a paper bound reprint from:

**Books on Demand**
ProQuest Information & Learning
(University of Microfilm International)
300 N. Zeeb Road
P.O. Box 1346, Ann Arbor
MI 48106-1346, USA
Tel.: 1-800-521-0600 (Customer Service)
http://wwwlib.umi.com/bod/basic

**Copyright** 2007 by editors and authors

Many books can be downloaded from the following
**Digital Library of Science**:
http://www.gallup.unm.edu/~smarandache/eBooks-otherformats.htm

ISBN 10: 1-59973-025-1
ISBN 13: 9781599730257
EAN: 1599730251

**Information for Authors**

Papers in electronic form are accepted. They can be e-mailed in Microsoft Word XP (or lower), WordPerfect 7.0 (or lower), LaTeX and PDF 6.0 or lower.

The submitted manuscripts may be in the format of remarks, conjectures, solved/unsolved or open new proposed problems, notes, articles, miscellaneous, etc. They must be original work and camera ready [typewritten/computerized, format: 8.5 x 11 inches (21,6 x 28 cm)]. They are not returned, hence we advise the authors to keep a copy.

The title of the paper should be writing with capital letters. The author's name has to apply in the middle of the line, near the title. References should be mentioned in the text by a number in square brackets and should be listed alphabetically. Current address followed by e-mail address should apply at the end of the paper, after the references.

The paper should have at the beginning an abstract, followed by the keywords.

All manuscripts are subject to anonymous review by three independent reviewers.

Every letter will be answered.

Each author will receive a free copy of this international book series.

**Contributing to Scientia Magna book series**

Authors of papers in science (mathematics, physics, engineering, philosophy, psychology, sociology, linguistics) should submit manuscripts to the main editor:

Prof. Dr. Zhang Wenpeng, Department of Mathematics, Northwest University, Xi'an, Shaanxi, P. R. China, E-mail: wpzhang@nwu.edu.cn.

**Associate Editors**

Dr. W. B. Vasantha Kandasamy, Department of Mathematics, Indian Institute of Technology, IIT Madras, Chennai - 600 036, Tamil Nadu, India.

Dr. Larissa Borissova and Dmitri Rabounski, Sirenevi boulevard 69-1-65, Moscow 105484, Russia.

Dr. Liu Huaning, Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China, E-mail: hnliu@nwu.edu.cn.

Prof. Yi Yuan, Research Center for Basic Science, Xi'an Jiaotong University, Xi'an, Shaanxi, P.R.China, E-mail: yuanyi@mail.xjtu.edu.cn.

Dr. Xu Zhefeng, Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China, E-mail: zfxu@nwu.edu.cn.

Dr. Zhang Tianping, College of Mathematics and Information Science, Shaanxi Normal University, Xi'an, Shaanxi, P.R.China, E-mail: tianpzhang@eyou.com.

# Preface

This issue of the journal is devoted to the proceedings of the third International Conference on Number Theory and Smarandache Problems held in Weinan during March 23-25, 2007. The organizers were myself and Professor Hailong Li from Weinan Teacher's University. The conference was supported by Weinan Teacher's University and there were more than 90 participants. We had five foreign guests, Professor F. Pappalardi and his daughter from Italy, Professor K. Chakraborty from India, Professor S. Kanemitsu and Y. Tanigawa from Japan. The conference was a great success and will give a strong impact on the development of number theory in general and Smarandache problems in particular. We hope this will become a tradition in our country and will continue to grow. And indeed we are planning to organize the fourth conference in coming March in Xianyang the famous old city which was the capital of Qin dynasty.

In the volume we assemble not only those papers which were presented at the conference but also those papers which were submitted later and are concerned with the Smarandache type problems or other mathematical problems.

There are a few papers which are not directly related to but should fall within the scope of Smarandache type problems. They are 1. J. Wang, Cube-free integers as sums of two squares; 2. G. Liu and H. Li, Recurrences for generalized Euler numbers; 3. X. Zhang and Y. Zhang, Sequences of numbers with alternate common differences.

Other papers are concerned with the number-theoretic Smarandache problems and will enrich the already rich stock of results on them. Readers can learn various techniques used in number theory and will get familiar with the beautiful identities and sharp asymptotic formulas obtained in the volume.

Researchers can download books on the Smarandache notions from the following open source Digital Library of Science:

www.gallup.unm.edu/~smarandache/eBooks-otherformats.htm.

Wenpeng Zhang

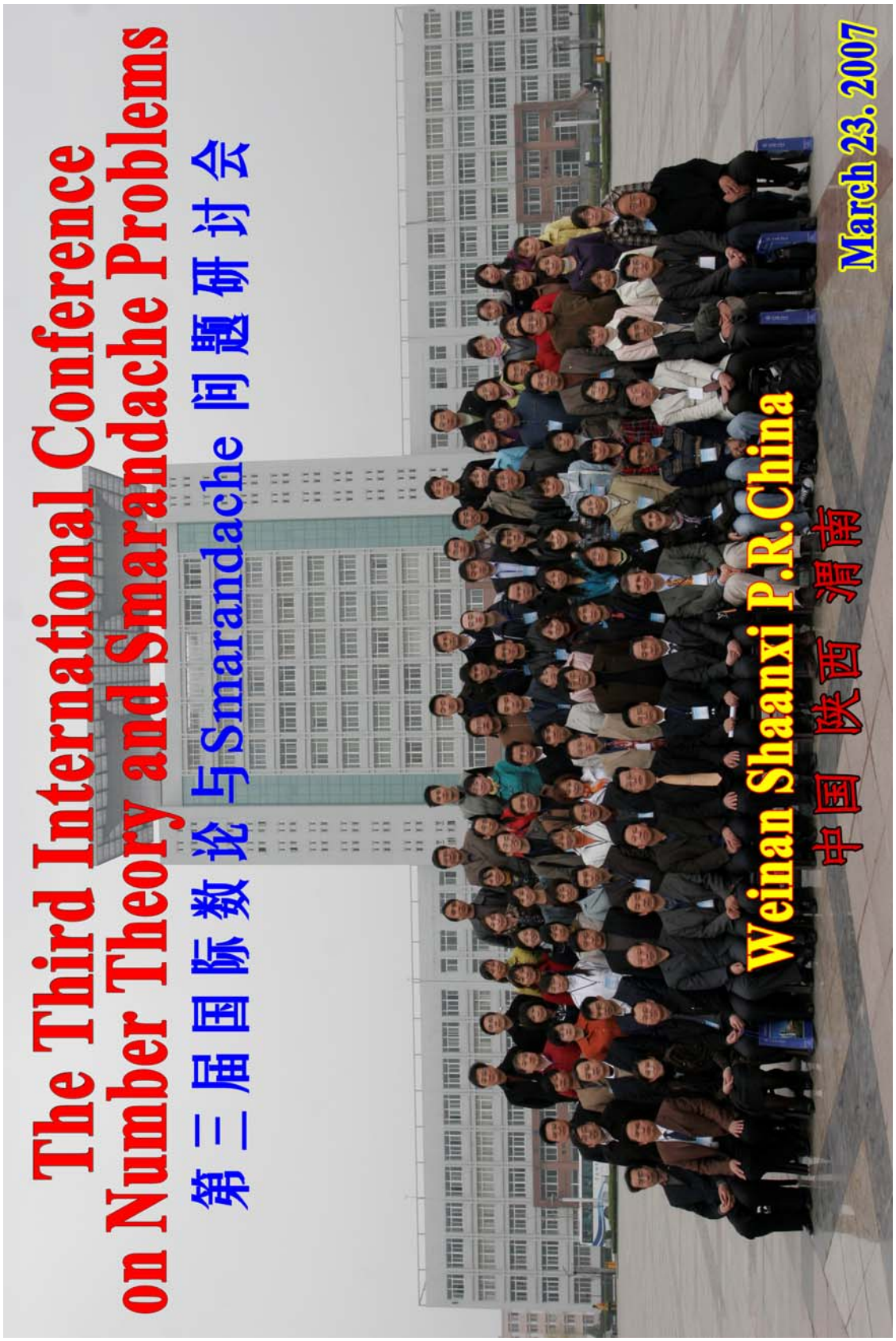
The Third International Conference on Number Theory
and Smarandache Problems

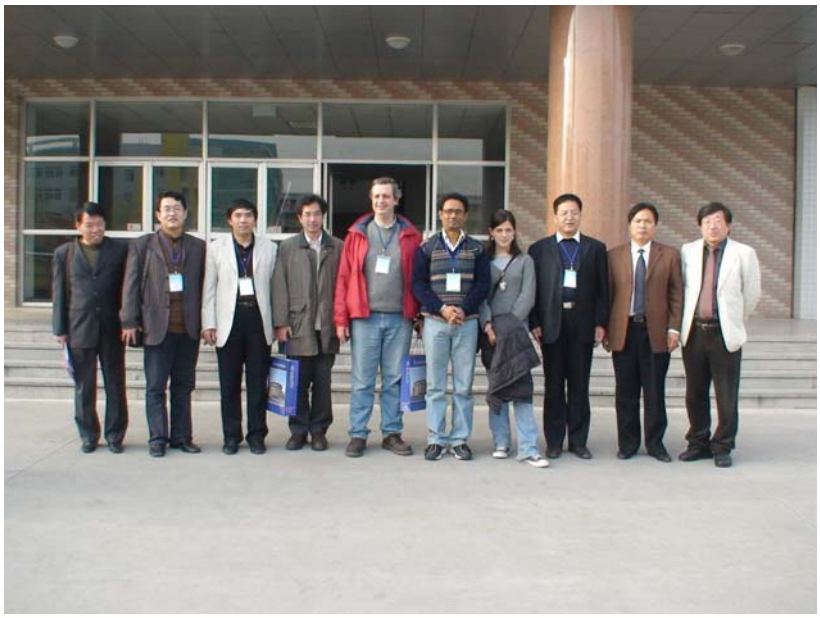
International conference on number theory and
Welcome to Weinan Teacher's University
热烈祝贺
隆重召开

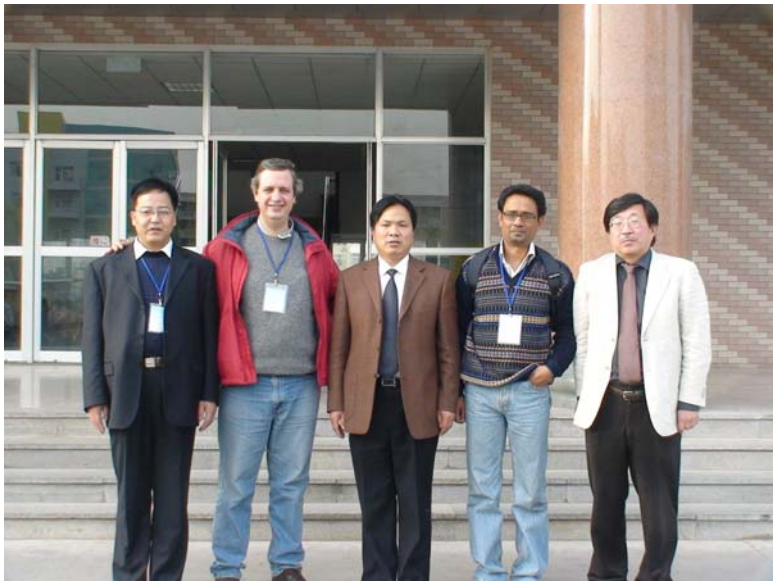
Welcome to Weinan Teacher's University
热烈祝贺
隆重召开

Welcome to Weinan Teacher's University
热烈祝贺

The participants of the Conference







Professor F. Pappalardi →

← Professor Wenguang Zhai

Professor Yoshio Tanigawa →

← Professor Wenpeng Zhang

← Professor Kalyan Chakraborty

Professor Shigeru Kanemitsu →

← Professor Linfan Mao

Professor Liu Guodong →

Professor Hailong Li →

← Professor Wansheng He

Professor Chaofeng Shi →

← Professor Yuan Yi

← Professor Li Gao

Professor Nianliang Wang →

← Dr. Ganglian Ren

Dr. Xiaowei Pan →

# Contents

**J. Wang :** Cube-free integers as sums of two squares 1

**G. Liu and H. Li :** Recurrences for generalized Euler numbers 9

**H. Li and Q. Yang :** Some properties of the LCM sequence 14

**M. Liu :** On the generalization of the primitive number function 18

**Z. Lv :** On the F. Smarandache LCM function and its mean value 22

**Q. Wu :** A conjecture involving the F. Smarandache LCM function 26

**S. Xue :** On the Smarandache dual function 29

**N. Yuan :** A new arithmetical function and its asymptotic formula 33

**A. Muktibodh, etc. :** Sequences of pyramidal numbers 39

**R. Zhang and S. Ma :** An efficient hybrid genetic algorithm for continuous optimization problems 46

**L. Mao :** An introduction to Smarandache multi-spaces and mathematical combinatorics 54

**M. Selariu :** Smarandache stepped functions 81

**X. Zhang and Y. Zhang :** Sequences of numbers with alternate common differences 93

**Y. Zhang :** On the near pseudo Smarandache function 98

**A. Muktibodh :** Smarandache mukti-squares 102

**Scientia Magna**
**Vol. 3 (2007), No. 1, 1-8**

# Cube-free integers as sums of two squares

Jia Wang

School of Mathematical Sciences, Shandong Normal University
Jinan, Shandong, P.R.China
Email: wangjia8233@126.com

Received December 1, 2006

**Abstract** Let $r(n)$ denote the number of representations of the integer $n$ as a sum of two squares, $q_3(n)$ be the characteristic function of the set of cube-free integers, and $P(x)$ the error term of the Gauss circle problem. Let $Q_3(x) := \sum_{n\leq x} q_3(n)r(n)$. In this paper we shall prove that if the estimate $P(x) = O(x^\theta)$ holds, then $Q_3(x+y) - Q_3(x) = Cy + O(yx^{-\varepsilon/2} + x^{\theta+\varepsilon})$, where C is a constant. In particular this asymptotic formula is true for $\theta = 131/416$.

**Keywords** Cube free integer; Gauss circle problem; Error term.

## §1. Introduction and Results

Let $r(n)$ denote number of representations of the integer $n$ by two squares. The famous Gauss circle problem is to study the upper bound of the error term $P(x) := \sum_{n\leq x} r(n) - \pi x$. Gauss first proved that $P(x) = O(x^{1/2})$. The exponent $1/2$ was later improved by many authors. The latest result is due to Huxley [2], who proved that

$$P(x) \ll x^{131/416}(\log x)^{26957/8320}. \tag{1}$$

For a survey of the history of the circle problem, see E. Krätzel [6].

Let $k \geq 2$ be a fixed integer and $q_k(n)$ denote the characteristic function of the $k$-free integers. The function $q_k(n)r(n)$ denote the number of representations of a $k$-free integer as a sum of two squares. Let

$$Q_k(x) := \sum_{n\leq x} q_k(n)r(n). \tag{2}$$

When $k = 2$, K.-H. Fischer [1] proved that

$$Q_2(x) = A_2 x + O(x^{1/2}\log x) \tag{3}$$

with

$$A_2 = \mathrm{Res}_{s=1}(1 + 2^{-s}) \prod_{p\in\mathcal{P}_1} (1 + 2p^{-s}),$$

where $\mathcal{P}_1$ denotes the set of all primes which are congruent to 1 modulo 4. The exponent 1/2 in (3) cannot be reduced with the present knowledge.

E. Krätzel [5] first studied the short interval case for $k = 2$. He proved that if

$$P(x) = O(x^\theta) \quad (\theta < 1/3) \tag{4}$$

and

$$D_3(x) := \sum_{n\le x} d_3(n) = x(c_1 \log^2 x + c_2 \log x + c_3) + O(x^\delta) \quad (\delta \le 1/2) \tag{5}$$

with some constants $c_1, c_2, c_3$, then the asymptotic formula

$$Q_2(x+y) - Q_2(x) = A_2 y + o(y) \tag{6}$$

holds for

$$x^{\frac{4-\delta(1+\theta)}{2(5-\theta-2\delta)}} \log^3 x \le y = o(x).$$

With the best present known estimates $\delta = 43/96 + \varepsilon$, $\theta = 131/416 + \varepsilon$(see Kolesnik [4] and Huxley [2], respectively), (5) is true for $y \ge x^{0.4501\cdots+\varepsilon}$. Since $\delta \ge 1/3$, $\theta \ge 1/4$, the limit of E. Krätzel's approach is $y \ge x^{43/98+\varepsilon}$. Wenguang Zhai [7] proved that if (4) is true, then (6) holds for $x^{\theta+\varepsilon} \le y \le x$.

When $k = 3$, it is easy to show that the asymptotic formula

$$Q_3(x) = Cx + O(x^{1/3} \log x) \tag{7}$$

holds, where $C$ is an computable constant. The exponent 1/3 can not be improved by the present method. In this short note, we shall prove the following

**Theorem.** Suppose (4) is true for some $1/4 < \theta < 1/3$, then we have

$$Q_3(x+y) - Q_3(x) = Cy + O(yx^{-\varepsilon/2} + x^{\theta+\varepsilon}). \tag{8}$$

where $C$ is a constant.

**Notations.** Throughout this paper, $\varepsilon$ always denotes a fixed sufficiently small positive constant. For any fixed integer $k$, $d_k(n)$ denotes the number of ways $n$ can be written as $k$ positive integers. $\zeta(s)$ is the Riemann zeta-function, and $L(s,\chi)$ is the Dirichlet L-function associated to the non-principal character $\chi \mod 4$.

## §2. Proof of the theorem

From

$$\sum_{n=1}^{\infty} \frac{r(n)}{n^s} = 4\zeta(s)L(s,\chi),$$

we can easily see that

$$\frac{r(n)}{4} = \sum_{d|n} \chi(d). \tag{9}$$

We have

$$\chi(n) = \begin{cases} 0, & n \equiv 0, 2(\mod 4), \\ 1, & n \equiv 1(\mod 4), \\ -1, & n \equiv 3(\mod 4). \end{cases} \tag{10}$$

Let

$$f(n) = \frac{1}{4} q_3(n) r(n), \tag{11}$$

It is easy to see that $f(n)$ is a multiplicative function. By (9), (10) and Euler product representation, we have

$$\begin{aligned} F(s) &:= \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{r(p)}{4p^s} + \frac{r(p^2)}{4p^{2s}}\right) \\ &= (1 + 2^{-s} + 4^{-s}) \prod_{p \in \mathcal{P}_1} (1 + 2p^{-s} + 3p^{-2s}) \prod_{p \in \mathcal{P}_3} (1 + p^{-2s}). \end{aligned} \tag{12}$$

Where $\mathcal{P}_1 = \{p | p \equiv 1(\mod 4)\}, \mathcal{P}_3 = \{p | p \equiv 3(\mod 4)\}$, respectively.

The following lemma plays a crucial role in the proof.

**Lemma 1.** Suppose $\Re s > 1$. Then we have

$$F(s) = \frac{\zeta(s) L(s, \chi) \zeta(4s) L^2(4s, \chi)}{\zeta^2(3s) L^2(3s, \chi)} M(s). \tag{13}$$

Where $M(s)$ is a certain Dirichlet series which is absolutely convergent for $\Re s > 1/5$.

**Proof.** By Tayler representation, we note for $|u| < 1/2$ , we have

$$1 + 2u + 3u^2 = (1 - u)^{-2} (1 - u^3)^4 (1 - u^4)^{-3} E(u). \tag{14}$$

with $E(u) = (1 + O(u^5))$. By the well-known Euler product formula, we have

$$\zeta(s) = (1 - 2^{-s})^{-1} \prod_{p \in \mathcal{P}_1} (1 - p^{-s})^{-1} \prod_{p \in \mathcal{P}_3} (1 - p^{-s})^{-1} \tag{15}$$

and

$$L(s, \chi) = \prod_{p \in \mathcal{P}_1} (1 - p^{-s})^{-1} \prod_{p \in \mathcal{P}_3} (1 + p^{-s})^{-1}. \tag{16}$$

So we have

$$\prod_{p \in \mathcal{P}_1} (1 - p^{-s})^{-2} = (1 - 2^{-s}) \zeta(s) L(s, \chi) \prod_{p \in \mathcal{P}_3} (1 - p^{-2s}). \tag{17}$$

From the above formulas we can rewrite (12) as

$$\begin{aligned} F(s) &= (1 + 2^{-s} + 4^{-s}) \prod_{p \in \mathcal{P}_1} (1 - p^{-s})^{-2} \prod_{p \in \mathcal{P}_1} (1 - p^{-3s})^4 \prod_{p \in \mathcal{P}_1} (1 - p^{-4s})^{-3} \prod_{p \in \mathcal{P}_3} (1 + p^{-2s}) \\ &\quad \cdot \prod_{p \in \mathcal{P}_1} E(p^{-s}). \end{aligned} \tag{18}$$

Note that

$$\begin{aligned}\prod_{p\in\mathcal{P}_1}(1-p^{-4s})^{-3} &= \zeta^2(4s)L^2(4s,\chi)(1-2^{-4s})^2\prod_{p\in\mathcal{P}_1}(1-p^{-4s})^2\prod_{p\in\mathcal{P}_3}(1-p^{-4s})^2\\ &\quad\cdot\prod_{p\in\mathcal{P}_1}(1-p^{-4s})^2\prod_{p\in\mathcal{P}_3}(1+p^{-4s})^2\prod_{p\in\mathcal{P}_1}(1-p^{-4s})^{-4}\prod_{p\in\mathcal{P}_1}(1-p^{-4s})\\ &= (1-2^{-4s})^2\zeta^2(4s)L^2(4s,\chi)\prod_{p\in\mathcal{P}_1}(1-p^{-4s})\prod_{p\in\mathcal{P}_3}(1-p^{-8s})^2 \qquad (19)\end{aligned}$$

and

$$\begin{aligned}\prod_{p\in\mathcal{P}_1}(1-p^{-3s})^4 &= \zeta^{-2}(3s)L^{-2}(3s,\chi)(1-2^{-3s})^{-2}\prod_{p\in\mathcal{P}_1}(1-p^{-3s})^{-2}\prod_{p\in\mathcal{P}_3}(1-p^{-3s})^{-2}\\ &\quad\cdot\prod_{p\in\mathcal{P}_1}(1-p^{-3s})^{-2}\prod_{p\in\mathcal{P}_3}(1+p^{-3s})^{-2}\prod_{p\in\mathcal{P}_1}(1-p^{-3s})^4\\ &= (1-2^{-3s})^{-2}\zeta^{-2}(3s)L^{-2}(3s,\chi)\prod_{p\in\mathcal{P}_3}(1-p^{-6s})^{-2}. \qquad (20)\end{aligned}$$

Combining(17), (19) and (20), we get

$$\begin{aligned}F(s) &= (1+2^{-s}+4^{-s})(1-2^{-s})\zeta(s)L(s,\chi)\prod_{p\in\mathcal{P}_3}(1-p^{-2s})\prod_{p\in\mathcal{P}_1}(1-p^{-4s})^{-3}\\ &\quad\cdot\prod_{p\in\mathcal{P}_1}(1-p^{-3s})^4\prod_{p\in\mathcal{P}_3}(1+p^{-2s})\prod_{p\in\mathcal{P}_1}E(p^{-s})\\ &= (1+2^{-s}+4^{-s})(1-2^{-s})\zeta(s)L(s,\chi)\prod_{p\in\mathcal{P}_1}(1-p^{-4s})^{-3}\prod_{p\in\mathcal{P}_1}(1-p^{-3s})^4\\ &\quad\cdot\prod_{p\in\mathcal{P}_3}(1-p^{-4s})\prod_{p\in\mathcal{P}_1}E(p^{-s})\\ &= \frac{(1+2^{-s}+4^{-s})(1-2^{-s})(1-2^{-4s})^2}{(1-2^{-3s})^2}\frac{\zeta(s)L(s,\chi)\zeta(4s)L^2(4s,\chi)}{\zeta^2(3s)L^2(3s,\chi)}\prod_{p\in\mathcal{P}_1}(1-p^{-4s})\\ &\quad\cdot\prod_{p\in\mathcal{P}_3}(1-p^{-8s})^2\prod_{p\in\mathcal{P}_3}(1-p^{-6s})^{-2}\prod_{p\in\mathcal{P}_3}(1-p^{-4s})\prod_{p\in\mathcal{P}_1}E(p^{-s})\\ &= \frac{\zeta(s)L(s,\chi)\zeta(4s)L^2(4s,\chi)}{\zeta^2(3s)L^2(3s,\chi)}M(s)\end{aligned}$$

with

$$M(s)=\frac{(1+2^{-s}+4^{-s})(1-2^{-s})(1-2^{-4s})}{(1-2^{-3s})^2}\prod_{p\in\mathcal{P}_3}(1-p^{-8s})^2\prod_{p\in\mathcal{P}_3}(1-p^{-6s})^{-2}\prod_{p\in\mathcal{P}_1}E(p^{-s}),$$

which has a Dirichlet series expansion, absolutely convergent for $\Re s > 1/5$.

By Lemma 1, for $\mathcal{R}s > 1$ we have,

$$F(s)=F_1(s)F_2(3s), \qquad (21)$$

where

$$F_1(s)=\sum_{n=1}^{\infty}\frac{f_1(n)}{n^s}=\zeta(s)L(s,\chi)\zeta(4s)L^2(4s,\chi)M(s), \qquad (22)$$

$$F_2(s)=\sum_{n=1}^{\infty}\frac{f_2(n)}{n^s}=\zeta^{-2}(s)L^{-2}(s,\chi). \qquad (23)$$

Then we have

**Lemma 2.** Suppose (4) holds, then

$$\sum_{n\leq x} f_1(n) = A_1 x + O(x^{\theta+\varepsilon}), \tag{24}$$

where

$$A_1 = \mathrm{Res}_{s=1} F_1(s).$$

**Proof.** We introduce the notation which we will use in this proof only.

$$M(s) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s},$$
$$\zeta(4s)L^2(4s,\chi)M(s) = \sum_{n=1}^{\infty} \frac{\beta(n)}{n^s},$$
$$\zeta(s)L^2(s,\chi) = \sum_{n=1}^{\infty} \frac{\gamma(n)}{n^s}.$$

Then we get

$$\beta(n) = \sum_{n=ml^4} \alpha(m)\gamma(l). \tag{25}$$

By Perron's formula we see that

$$\sum_{n\leq x} |\alpha(n)| \ll x^{1/5+\varepsilon}. \tag{26}$$

and

$$\gamma(n) = \sum_{n=n_1n_2n_3} \chi(n_2)\chi(n_3) \ll d_3(n), \tag{27}$$

Combining (26), (27), (5) and partial summation, we have

$$\begin{aligned}\sum_{n\leq x} |\beta(n)| &\ll \sum_{ml^4\leq x} |\alpha(m)| d_3(l)\\ &= \sum_{m\leq x} |\alpha(m)| D_3(x^{1/4}m^{-1/4})\\ &\ll x^{1/4}\log^2 x \sum_{m\leq x} m^{-1/4}|\alpha(m)|\\ &\ll x^{1/4}\log^2 x.\end{aligned} \tag{28}$$

Also, we have

$$f_1(n) = \sum_{n=hd} \frac{1}{4} r(h)\beta(d). \tag{29}$$

Now we may appeal Ivić([3], Theorem 14.1), combining(3), (28), (29) to conclude (24).

$$\begin{aligned}\sum_{n\le x} f_1(n) &= \sum_{h\le x^{\varepsilon}} \frac{1}{4} r(h) \sum_{d\le x/h} \beta(d) + \sum_{d\le x^{1-\varepsilon}} \beta(d) \sum_{h\le x/d} \frac{1}{4} r(h) \\ &\quad - \left(\sum_{h\le x^{\varepsilon}} \frac{1}{4} r(h)\right)\left(\sum_{d\le x^{1-\varepsilon}} \beta(d)\right) \\ &= \frac{1}{4}\pi x \sum_{d\le x^{1-\varepsilon}} \beta(d) d^{-1} + O\left(x^{\theta} \sum_{d\le x^{1-\varepsilon}} |\beta(d)| d^{-\theta}\right) \\ &\quad + O\left(x^{\frac{1}{4}} \log^2 x \sum_{h\le x^{\varepsilon}} r(h) h^{-\frac{1}{4}}\right) + O\left(x^{\varepsilon}\cdot x^{\frac{1}{4}-\frac{\varepsilon}{4}}\right) \\ &= \frac{1}{4}\pi c x + O(x^{\theta}). \end{aligned} \tag{30}$$

where $c = \int_1^{\infty} \left(\sum_{d\le u} \beta(d)\right) u^{-2} du$.

Since the generating function for $f_1(n)$ is $F_1(s)$, having a simple pole at $s = 1$, then the main term must coincide in view of Perron's formula:

$$\frac{1}{4}\pi c = A_1 = \mathrm{Res}_{s=1} F_1(s).$$

The following lemma is prepared for estimating the error term.

**Lemma 3.** Let $k \ge 2$ be a fixed integer, $1 < y \le x$ be large real numbers and

$$\mathcal{A}(x, y; k, \varepsilon) := \sum_{\substack{x < nm^k \le x+y \\ m > x^{\varepsilon}}} 1.$$

Then we have

$$\mathcal{A}(x, y; k, \varepsilon) \le y x^{-\varepsilon} + x^{1/4}. \tag{31}$$

**Proof.** This is Lemma 3 of Zhai [7].

Now we will finish proving our Theorem. From (23), we get

$$f_2(n) = \sum_{n = l_1 l_2 l_3 l_4} \mu(l_1)\mu(l_2)\mu(l_3)\mu(l_4)\chi(l_3)\chi(l_4) \ll d_4(n)$$

and

$$\sum_{n\le x} f_2(n) \ll \sum_{n\le x} d_4(n) \ll x \log^3 x. \tag{32}$$

From (21), we get

$$f(n) = \sum_{n = n_1 n_2^3} f_1(n_1) f_2(n_2). \tag{33}$$

By (2.25), we have

$$\begin{aligned} Q_3(x+y)-Q_3(x) &= 4\sum_{x<n\le x+y} f(n) \\ &= 4\sum_{\substack{x<n_1n_2^3\le x+y\\ n_2\le x^{\varepsilon}}} f_1(n_1)f_2(n_2)+4\sum_{\substack{x<n_1n_2^3\le x+y\\ n_2> x^{\varepsilon}}} f_1(n_1)f_2(n_2) \\ &= 4\sum_1+O\Big(\sum_2\Big). \end{aligned} \tag{34}$$

where

$$\sum_1=\sum_{n_2\le x^{\varepsilon}} f_2(n_2)\sum_{\frac{x}{n_2^3}<n_1\le\frac{x+y}{n_2^3}} f_1(n_1),\qquad \sum_2=\sum_{\substack{x<n_1n_2^3\le x+y\\ n_2> x^{\varepsilon}}} |f_1(n_1)f_2(n_2)|.$$

First, we will deal with $\sum_1$. By Lemma2, we have

$$\begin{aligned} \sum_1 &= A_1y\sum_{n_2\le x^{\varepsilon}} f_2(n_2)n_2^{-3}+O\Big(x^{\theta}\sum_{n_2\le x^{\varepsilon}} f_2(n_2)n_2^{-3\theta}\Big) \\ &= A_1y\sum_{n=1}^{\infty} f_2(n_2)n_2^{-3}+O\left(yx^{-\varepsilon/2}+x^{\theta+\varepsilon}\right). \end{aligned} \tag{35}$$

Now we deal with $\sum_2$. By Lemma 3, and noting

$$f_1(n)\ll n^{\varepsilon^2},\quad f_2(n)\ll n^{\varepsilon^2},$$

we have

$$\sum_2\ll\sum_{\substack{x<n_1n_2^3\le x+y\\ n_2> x^{\varepsilon}}}(n_1n_2)^{\varepsilon^2}=x^{\varepsilon^2}\mathcal{A}(x,y;3,\varepsilon)\ll yx^{-\varepsilon/2}+x^{1/4+\varepsilon}. \tag{36}$$

Finally, combining (34)-(36), we have

$$\begin{aligned} Q_3(x+y)-Q_3(x) &= 4A_1y\sum_{n_2=1}^{\infty}\frac{f_2(n_2)}{n_2^3}+O(yx^{-\varepsilon/2}+x^{\theta+\varepsilon}) \\ &= Cy+O(yx^{-\varepsilon/2}+x^{\theta+\varepsilon}). \end{aligned}$$

Since the generating function for $f(n)$ is $F(s)$, so the main term

$$A_1\sum_{n_2=1}^{\infty}\frac{f_2(n_2)}{n_2^3}$$

must coincides with $\mathrm{Res}_{s=1}F(s)$. As a result,

$$C=4A_1\sum_{n_2=1}^{\infty}\frac{f_2(n_2)}{n_2^3}=4A=4\mathrm{Res}_{s=1}F(s).$$

# References

[1] K.-H. Fischer, Über die Anzahl der Gitterpunkte auf Kreisen mit quadratfreien Radienquadraten, Arch. Math., **33**(1979), 150-154.

[2] M. N. Huxley, Exponential sums and Lattice points III, Proc. London Math. Soc., **87**(3)(2003), 591-609.

[3] A. Ivić, The Riemann zeta-function, John Wiley & Sons, New York.

[4] G. Kolesnik, On the estimation of multiple exponential sums, in Recent Progress in Analytic Number Theory, Symposium Durham 1979(Vol.1), Academic, London, 231-246.

[5] E. Krätzel, Squarefree numbers as sums of two squares, Arch.Math. **39**(1982), 28-31.

[6] E. Krätzel, Lattice points, Deutsch. Verlag Wiss. Berlin, 1988.

[7] Wenguang Zhai, Square-free integers as sums of two squares, Number Theory: Tradition and Modernization, Springer, New York, 2006, 219-227.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 9-13**

# Recurrences for generalized Euler numbers[1]

Guodong Liu† and Hui Li‡

†. Department of Mathematics, Huizhou University,
Huizhou, Guangdong, P. R. China
e-mail: gdliu@pub.huizhou.gd.cn.

‡. Department of Mathematics, Jiaying University,
Meizhou, Guangdong, P.R. China

Received Jan. 23, 2007

**Abstract** In this paper, we establish some recurrence formulas for generalized Euler numbers.

**Keywords** Euler numbers, generalized Euler numbers, Bernoulli numbers, recurrence formula.

## §1. Introduction and results

For an integer $k$, the generalized Euler numbers $E_{2n}^{(k)}$ and the generalized Bernoulli numbers $B_n^{(k)}$ are defined by the following generating functions (see, for details, [1], [2], [3] and [4]):

$$(\sec t)^k = \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(k)} \frac{t^{2n}}{(2n)!} \tag{1}$$

and

$$\left(\frac{t}{e^t - 1}\right)^k = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} \tag{2}$$

respectively. Clearly, we have

$$E_{2n}^{(1)} = E_{2n} \qquad \text{and} \qquad B_n^{(1)} = B_n \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) \tag{3}$$

in terms of the classical Euler numbers $E_{2n}$ and the classical Bernoulli numbers $B_n$, $\mathbb{N}$ being the set of positive integers. The Euler numbers $E_{2n}$ and the Bernoulli numbers $B_n$ satisfy

$$E_0 = 1, \qquad E_{2n} = -\sum_{k=0}^{n-1} \binom{2n}{2k} E_{2k} \tag{4}$$

and

$$B_0 = 1, \qquad B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k. \tag{5}$$

By (1) and (2), we have

$$B_{2n} = \frac{2n}{2^{2n}(2^{2n} - 1)} E_{2n-2}^{(2)} \qquad (n \in \mathbb{N}). \tag{6}$$

[1]This work is supported by Guangdong Provincial Natural Science Foundation of China (05005928).

Numerous interesting (and useful) properties and relationships involving each of these families of numbers can be found in many books and tables (see [5], [6] and [7]). The main purpose of this paper is to establish some recurrence formulas for generalized Euler numbers. That is, we shall prove the following main conclusion.

**Theorem 1.** Let $n \in \mathbb{N}, k \in \mathbb{N}_0$. Then

$$E_{2n}^{(2k+1)} = -\frac{1}{2^{2k}} \sum_{i=0}^{k} \binom{2k+1}{i} \sum_{j=0}^{n-1} \binom{2n}{2j} (2k+1-2i)^{2n-2j} E_{2j}^{(2k+1)}. \tag{7}$$

**Remark 1.** Setting $k = 0$ in (7), we immediately obtain (4).

**Theorem 2.** Let $n \in \mathbb{N}, k \in \mathbb{N}_0$. Then

$$E_{2n}^{(2k+2)} = -\frac{1}{2^{2k+1}} \sum_{i=0}^{k} \binom{2k+2}{i} \sum_{j=0}^{n-1} \binom{2n}{2j} (2k+2-2i)^{2n-2j} E_{2j}^{(2k+2)}. \tag{8}$$

**Remark 2.** Setting $k = 0$ in (8), we can get

$$E_{2n}^{(2)} = -\frac{1}{2} \sum_{j=0}^{n-1} \binom{2n}{2j} 2^{2n-2j} E_{2j}^{(2)} \quad (n \in \mathbb{N}). \tag{9}$$

By (9) and (6), we have

$$B_{2n} = \frac{n}{2(1-2^{2n})} \sum_{j=1}^{n-1} \binom{2n-2}{2j-2} \frac{(2^{2j}-1)}{j} B_{2j} \quad (n \in \mathbb{N} \setminus \{1\}). \tag{10}$$

## §2. Some lemmas

**Lemma 1.** Let $k \in \mathbb{N}_0$. Then

$$2^{2k} \cos^{2k+1} x = \sum_{j=0}^{k} \binom{2k+1}{j} \cos(2k+1-2j)x. \tag{11}$$

**Proof.**

$$\begin{aligned} 2^{2k+1} \cos^{2k+1} x &= (e^{ix} + e^{-ix})^{2k+1} = \sum_{j=0}^{2k+1} \binom{2k+1}{j} e^{(2k+1-2j)ix} \\ &= \sum_{j=0}^{k} \binom{2k+1}{j} e^{(2k+1-2j)ix} + \sum_{j=k+1}^{2k+1} \binom{2k+1}{j} e^{(2k+1-2j)ix} \\ &= \sum_{j=0}^{k} \binom{2k+1}{j} e^{(2k+1-2j)ix} + \sum_{j=0}^{k} \binom{2k+1}{j} e^{-(2k+1-2j)ix} \\ &= 2 \sum_{j=0}^{k} \binom{2k+1}{j} \cos(2k+1-2j)x. \end{aligned} \tag{12}$$

i.e.

$$2^{2k} \cos^{2k+1} x = \sum_{j=0}^{k} \binom{2k+1}{j} \cos(2k+1-2j)x. \tag{13}$$

**Lemma 2.** Let $k \geq 0$ be integers, then

$$2^{2k+1}\cos^{2k+2}x = \sum_{j=0}^{k}\binom{2k+2}{j}\cos(2k+2-2j)x + \binom{2k+1}{k}. \tag{14}$$

**Proof.**

$$\begin{aligned}
2^{2k+2}\cos^{2k+2}x &= (e^{ix}+e^{-ix})^{2k+2} = \sum_{j=0}^{2k+2}\binom{2k+2}{j}e^{(2k+2-2j)ix}\\
&= \sum_{j=0}^{k}\binom{2k+2}{j}e^{(2k+2-2j)ix} + \sum_{j=k+1}^{2k+2}\binom{2k+2}{j}e^{(2k+2-2j)ix}\\
&= \sum_{j=0}^{k}\binom{2k+2}{j}e^{(2k+2-2j)ix} + \sum_{j=0}^{k+1}\binom{2k+2}{j}e^{-(2k+2-2j)ix}\\
&= 2\sum_{j=0}^{k}\binom{2k+2}{j}\cos(2k+2-2j)x + \binom{2k+2}{k+1}\\
&= 2\sum_{j=0}^{k}\binom{2k+2}{j}\cos(2k+2-2j)x + 2\binom{2k+1}{k}.
\end{aligned} \tag{15}$$

i.e.

$$2^{2k+1}\cos^{2k+2}x = \sum_{j=0}^{k}\binom{2k+2}{j}\cos(2k+2-2j)x + \binom{2k+1}{k}. \tag{16}$$

**Remark 3.** Taking $x = 0$ in Lemma 1 and Lemma 2, we can get

$$\sum_{i=0}^{k}\binom{2k+1}{i} = 2^{2k} \tag{17}$$

and

$$\sum_{i=0}^{k}\binom{2k+2}{i} + \binom{2k+1}{k} = 2^{2k+1}. \tag{18}$$

## §3. Proof of the theorems

**Proof of Theorem 1.** By Lemma 1 and (1), we have

$$\begin{aligned}
2^{2k} &= \sum_{n=0}^{\infty}(-1)^n E_{2n}^{(2k+1)}\frac{x^{2n}}{(2n)!}\sum_{i=0}^{k}\binom{2k+1}{i}\sum_{n=0}^{\infty}(-1)^n(2k+1-2i)^{2n}\frac{x^{2n}}{(2n)!}\\
&= \sum_{i=0}^{k}\binom{2k+1}{i}\sum_{n=0}^{\infty}(-1)^n\sum_{j=0}^{n}\binom{2n}{2j}(2k+1-2i)^{2n-2j}E_{2j}^{(2k+1)}\frac{x^{2n}}{(2n)!}.
\end{aligned} \tag{19}$$

and comparing the coefficient of $x^{2n}$ on both sides of (19), we get

$$\sum_{i=0}^{k}\binom{2k+1}{i}\sum_{j=0}^{n}\binom{2n}{2j}(2k+1-2i)^{2n-2j}E_{2j}^{(2k+1)} = 0, \quad (n \in \mathbb{N}) \tag{20}$$

i.e.

$$\sum_{i=0}^{k}\binom{2k+1}{i}\sum_{j=0}^{n-1}\binom{2n}{2j}(2k+1-2i)^{2n-2j}E_{2j}^{(2k+1)}+\sum_{i=0}^{k}\binom{2k+1}{i}E_{2n}^{(2k+1)}=0. \tag{21}$$

By (3.3) and (2.7), we immediately obtain Theorem 1. This completes the proof of Theorem 1.

**Proof of Theorem 2.** By Lemma 2 and (1), we have

$$\begin{aligned}2^{2k+1}&=\sum_{n=0}^{\infty}(-1)^nE_{2n}^{(2k+2)}\frac{x^{2n}}{(2n)!}\sum_{i=0}^{k}\binom{2k+2}{i}\sum_{n=0}^{\infty}(-1)^n(2k+2-2i)^{2n}\frac{x^{2n}}{(2n)!}\\&+\binom{2k+2}{i}\sum_{n=0}^{\infty}(-1)^nE_{2n}^{(2k+2)}\frac{x^{2n}}{(2n)!}\\&=\sum_{i=0}^{k}\binom{2k+2}{i}\sum_{n=0}^{\infty}(-1)^n\sum_{j=0}^{n}\binom{2n}{2j}(2k+2-2i)^{2n-2j}E_{2j}^{(2k+2)}\frac{x^{2n}}{(2n)!}\\&+\binom{2k+2}{i}\sum_{n=0}^{\infty}(-1)^nE_{2n}^{(2k+2)}\frac{x^{2n}}{(2n)!}.\end{aligned} \tag{22}$$

and comparing the coefficient of $x^{2n}$ on both sides of (22), we get

$$\sum_{i=0}^{k}\binom{2k+2}{i}\sum_{j=0}^{n}\binom{2n}{2j}(2k+2-2i)^{2n-2j}E_{2j}^{(2k+2)}+\binom{2k+1}{k}E_{2n}^{(2k+2)}=0,\quad(n\in\mathbb{N}) \tag{23}$$

i.e.

$$\begin{aligned}&\sum_{i=0}^{k}\binom{2k+2}{i}\sum_{j=0}^{n-1}\binom{2n}{2j}(2k+2-2i)^{2n-2j}E_{2j}^{(2k+2)}+\left(\sum_{i=0}^{k}\binom{2k+2}{i}+\binom{2k+1}{k}\right)E_{2n}^{(2k+2)}\\&=0.\end{aligned} \tag{24}$$

By (24) and (18), we immediately obtain Theorem 2. This completes the proof of Theorem 2.

# References

[1] Liu, G. D., Srivastava, H. M., Explicit formulas for the Nörlund polynomials $B_n^{(x)}$ and $b_n^{(x)}$, Comput. Math. Appl. **51**(2006), 1377-1384.

[2] Liu, G. D., Zhang, W. P., Applications of an explicit formula for the generalized Euler numbers, Acta Math. Sinica (English Series) (Chinese), to appear.

[3] Liu, G. D., Congruences for higher-order Euler numbers, Proc. Japan Acad. (Ser.A), **82**(2006), 30-33.

[4] Liu, G. D., Summation and recurrence formula involving the central factorial numbers and zeta function, Appl. Math. Comput. **149**(2004), 175-186.

[5] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F. G., Higher Transcendental Functions, Vol. I, McGraw-Hill Book Company, 1953.

[6] Luke, Y. L., The Special Functions and Their Approximations, Vol. I, Academic Press, New York and London, 1969.

[7] Srivastava, H. M., Choi, J., Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 14-17**

# Some properties of the LCM sequence

Hailong Li and Qianli Yang

Department of Mathematics, Weinan Teacher's College
Weinan, Shaanxi, P.R.China

Received Dec. 1, 2006

**Abstract** The main purpose of this paper is using the elementary method to study the properties of the Smarandache LCM sequence, and give some interesting identities.

**Keywords** Smarandache LCM sequence, elementary method, identities.

## §1. Introduction and results

For any positive integer $n$, we define $L(n)$ as the Least Common Multiply (LCM) of the natural number from 1 to $n$. That is,

$$L(n) = [1,\ 2,\ \cdots,\ n].$$

The Smarandache Least Common Multiply Sequence is defined by:

$$\text{SLS} \longrightarrow L(1),\ L(2),\ L(3),\ \cdots,\ L(n),\ L(n+1),\ \cdots.$$

For example, the first few values in the sequence $\{L(n)\}$ are: $L(1) = 1$, $L(2) = 2$, $L(3) = 6$, $L(4) = 12$, $L(5) = 60$, $L(6) = 60$, $L(7) = 420$, $L(8) = 840$, $L(9) = 2520$, $L(10) = 2520$, $\cdots$.

About the elementary arithmetical properties of $L(n)$, there are many results in elementary number theory text books (See references [2] and [3]), such as:

$$[a,b] = \frac{ab}{(a,b)} \quad \text{and} \quad [a,b,c] = \frac{abc\cdot(a,b,c)}{(a,b)(b,c)(c,a)},$$

where $(a_1, a_2, \cdots, a_k)$ denotes the Greatest Common Divisor of $a_1$, $a_2$, $\cdots$, $a_{k-1}$ and $a_k$.

Recently, Pan Xiaowei [4] studied the deeply arithmetical properties of L($n$), and proved that for any positive integer $n > 2$, we have the asymptotic formula:

$$\left(\frac{L(n^2)}{\prod\limits_{p\le n^2} p}\right)^{\frac{1}{n}} = e + O\left(\exp\left(-c\frac{(\ln n)^{\frac{3}{5}}}{(\ln\ln n)^{\frac{1}{5}}}\right)\right),$$

where $c$ is a positive constant, and $\prod\limits_{p\le n^2}$ denotes the product over all primes $p \le n^2$.

In this paper, we shall use the elementary method to study the calculating problem of $L(n)$, and give an exact calculating formula for it. That is, we shall prove the following:

**Theorem 1.** For any positive integer $n > 1$, we have the calculating formula

$$L(n) = \exp\left(\sum_{k=1}^{\infty} \theta\left(n^{\frac{1}{k}}\right)\right) = \exp\left(\sum_{k\le n} \Lambda(k)\right),$$

where $\exp(y) = e^y$, $\theta(x) = \sum_{p\le x} \ln p$, $\sum_{p\le x}$ denotes the summation over all primes $p \le x$, and $\Lambda(n)$ is the Mangoldt function defined as follows:

$$\Lambda(n) = \begin{cases} \ln p, & \text{if } n = p^{\alpha},\ p \text{ be a prime, and } \alpha \text{ be a positive integer;} \\ 0, & \text{otherwise.} \end{cases}$$

Now let $d(n)$ denotes the Dirichlet divisor function, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of $n$ into prime powers. We define the function $\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_k$. Then we have the following:

**Theorem 2.** For any positive integer $n > 1$, we have the calculating formula

$$\Omega\left(L(n)\right) = \sum_{k=1}^{\infty} \pi\left(n^{\frac{1}{k}}\right).$$

**Theorem 3.** For all positive integer $n \ge 2$, we also have

$$d\left(L(n)\right) = \exp\left(\sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k}\right) \pi\left(n^{\frac{1}{k}}\right)\right),$$

where $\exp(y) = e^y$ and $\pi(x) = \sum_{p\le x} 1$.

From these theorems and the famous Prime Theorem we may immediately deduce the following two corollaries:

**Corollary 1.** Under the notations of the above, we have

$$\lim_{n\to\infty} [L(n)]^{\frac{1}{n}} = e \quad \text{and} \quad \lim_{n\to\infty} \left[d\left(L(n)\right)\right]^{\frac{1}{\Omega(L(n))}} = 2,$$

where $e = 2.718281828459\cdots$ is a constant.

**Corollary 2.** For any integer $n > 1$, we have the asymptotic formula

$$\Omega\left(L(n)\right) = \frac{n}{\ln n} + O\left(\frac{n}{\ln^2 n}\right).$$

## §2. Proof of the theorems

In this section, we shall complete the proof of these theorems. First we prove Theorem 1. Let

$$L(n) = [1,\ 2,\ \cdots,\ n] = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} = \prod_{p\le n} p^{\alpha(p)} \tag{1}$$

be the factorization of $L(n)$ into prime powers. Then for each $1 \le i \le s$, there exists a positive integer $1 < k \le n$ such that $p_i^{\alpha_i} \parallel k$. So from (1) we have

$$\begin{aligned} L(n) &= [1,\ 2,\ \cdots,\ n] = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} = \exp\left(\sum_{t=1}^{s} \alpha_t \ln p_t\right) = \exp\left(\sum_{p\le n} \alpha(p)\ln p\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \sum_{n^{\frac{1}{k+1}}<p\le n^{\frac{1}{k}}} \alpha(p)\ln p\right). \end{aligned} \tag{2}$$

Note that if $n^{\frac{1}{k+1}} < p \le n^{\frac{1}{k}}$, then $p^k \le n$, $p^{k+1} > n$ and $\alpha(p) = k$. So from (2) we have

$$\begin{aligned} L(n) &= \exp\left(\sum_{k=1}^{\infty} \sum_{n^{\frac{1}{k+1}}<p\le n^{\frac{1}{k}}} k\cdot\ln p\right) \\ &= \exp\left(\sum_{k=1}^{\infty} k\left(\sum_{n^{\frac{1}{k+1}}<p\le n^{\frac{1}{k}}} \ln p\right)\right) \\ &= \exp\left(\sum_{k=1}^{\infty} k\left[\theta\left(n^{\frac{1}{k}}\right) - \theta\left(n^{\frac{1}{k+1}}\right)\right]\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \left[k\theta\left(n^{\frac{1}{k}}\right) - (k+1)\theta\left(n^{\frac{1}{k+1}}\right) + \theta\left(n^{\frac{1}{k+1}}\right)\right]\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \theta\left(n^{\frac{1}{k}}\right)\right) = \exp\left(\sum_{k\le n} \Lambda(k)\right), \end{aligned}$$

where $\theta(x) = \sum_{p\le x} \ln p$, and $\Lambda(n)$ is the Mangoldt function. This proves Theorem 1.

Now we prove Theorem 2. In fact from the definition of $\Omega(n)$ and the method of proving Theorem 1 we have

$$\begin{aligned} \Omega\left(L(n)\right) &= \sum_{p\le n} \alpha(p) = \sum_{k=1}^{\infty} \sum_{n^{\frac{1}{k+1}}<p\le n^{\frac{1}{k}}} \alpha(p) = \sum_{k=1}^{\infty} \sum_{n^{\frac{1}{k+1}}<p\le n^{\frac{1}{k}}} k \\ &= \sum_{k=1}^{\infty} k\left(\sum_{n^{\frac{1}{k+1}}<p\le n^{\frac{1}{k}}} 1\right) \\ &= \sum_{k=1}^{\infty} k\left[\pi\left(n^{\frac{1}{k}}\right) - \pi\left(n^{\frac{1}{k+1}}\right)\right] \\ &= \sum_{k=1}^{\infty} \left[k\pi\left(n^{\frac{1}{k}}\right) - (k+1)\pi\left(n^{\frac{1}{k+1}}\right) + \pi\left(n^{\frac{1}{k+1}}\right)\right] \\ &= \sum_{k=1}^{\infty} \pi\left(n^{\frac{1}{k}}\right), \end{aligned}$$

where $\pi(x) = \sum_{p\le x} 1$. This proves Theorem 2.

Note that the definition of the Dirichlet divisor function $d(n)$ we have

$$\begin{aligned}
d\left(L(n)\right) &= \prod_{p\leq n}\left(\alpha(p)+1\right)=\exp\left(\sum_{p\leq n}\ln[\alpha(p)+1]\right)\\
&= \exp\left(\sum_{k=1}^{\infty}\sum_{n^{\frac{1}{k+1}}<p\leq n^{\frac{1}{k}}}\ln[\alpha(p)+1]\right)\\
&= \exp\left(\sum_{k=1}^{\infty}\sum_{n^{\frac{1}{k+1}}<p\leq n^{\frac{1}{k}}}\ln(k+1)\right)\\
&= \exp\left(\sum_{k=1}^{\infty}\ln(k+1)\sum_{n^{\frac{1}{k+1}}<p\leq n^{\frac{1}{k}}}1\right)\\
&= \exp\left(\sum_{k=1}^{\infty}\ln(k+1)\left[\pi\left(n^{\frac{1}{k}}\right)-\pi\left(n^{\frac{1}{k+1}}\right)\right]\right)\\
&= \exp\left(\sum_{k=1}^{\infty}\left[\ln(k)\pi\left(n^{\frac{1}{k}}\right)-\ln(k+1)\pi\left(n^{\frac{1}{k+1}}\right)+\ln\left(1+\frac{1}{k}\right)\pi\left(n^{\frac{1}{k}}\right)\right]\right)\\
&= \exp\left(\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k}\right)\pi\left(n^{\frac{1}{k}}\right)\right).
\end{aligned}$$

This completes the proof of Theorem 3.

Corollary 1 and Corollary 2 follows from our theorems and the asymptotic formulae:

$$\theta(x)=\sum_{p\leq x}\ln p=x+O\left(x\exp\left(-c\frac{(\ln x)^{\frac{3}{5}}}{(\ln\ln x)^{\frac{1}{5}}}\right)\right)\quad\text{and}\quad\pi(x)=\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right),$$

where $c>0$ is a constant. These formulae can be found in reference [5].

# References

[1] Amarnth Murthy, Generalized Partitions and New Ideas On Number Theory and Smarandache Sequences, Hexis, 2005, 20-22.

[2] Tom M.Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

[3] Pan Chengdong and Pan Chengbiao, Elementary Number Theory, Beijing University Press, Beijing, 1992.

[4] Pan Xiaowei, A new limit theorem involving the Smarandache LCM sequence, Scientia Magna, **2**(2006), No.2, 20-23.

[5] Pan Chengdong and Pan Chengbiao, Foundation of Analytic Number Theory, Science Press, Beijing, 1999, 202-205.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 18-21**

# On the generalization of the primitive number function

Miaohua Liu

Department of Mthematics, Northwest University
Xi'an, Shaanxi, P.R.China

Received Aug. 25, 2006

**Abstract** Let $k$ be any fixed positive integer, $n$ be any positive integer, $S_k(n)$ denotes the smallest positive integer $m$ such that $m!$ is divisible by $k^n$. In this paper, we use the elementary methods to study the asymptotic properties of $S_k(n)$, and give an interesting asymptotic formula for it.

**Keywords** F.Smarandache problem, primitive numbers, asymptotic formula.

## §1. Introduction

For any fixed positive integer $k > 1$ and any positive integer $n$, we define function $S_k(n)$ as the smallest positive integer $m$ such that $k^n \mid m!$. That is,

$$S_k(n) = \min\{m : m \in N, k^n \mid m!\}.$$

For example, $S_4(1) = 4$, $S_4(2) = 6$, $S_4(3) = 8$, $S_4(4) = 10$, $S_4(5) = 12, \cdots$. In problem 49 of book [1], Professor F.Smarandache asked us to study the properties of the sequence $\{S_p(n)\}$, where $p$ is a prime. The problem is interesting because it can help us to calculate the Smarandache function. About this problem, many scholars have shown their interest on it, see [2], [3], [4] and [5]. For example, professor Zhang Wenpeng and Liu Duansen had studied the asymptotic properties of $S_p(n)$ in reference [2], and give an interesting asymptotic formula:

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p}\ln n\right).$$

Yi Yuan [3] had studied the mean value distribution property of $|S_p(n+1) - S_p(n)|$, and obtained the following asymptotic formula: for any real number $x \geq 2$, let $p$ be a prime and $n$ be any positive integer, then

$$\frac{1}{p}\sum_{n\leq x}|S_p(n+1) - S_p(n)| = x\left(1 - \frac{1}{p}\right) + O\left(\frac{\ln x}{\ln p}\right).$$

Xu Zhefeng [4] had studied the relationship between the Riemann zeta-function and an infinite series involving $S_p(n)$, and obtained some interesting identities and asymptotic formula

for $S_p(n)$. That is, for any prime $p$ and complex number $s$ with Re $s > 1$, we have the identity:

$$\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

where $\zeta(s)$ is the Riemann zeta-function.

And let $p$ be any fixed prime, then for any real number $x \geq 1$,

$$\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1}\left(\ln x + \gamma + \frac{p \ln p}{p-1}\right) + O\left(x^{-\frac{1}{2}+\varepsilon}\right),$$

where $\gamma$ is the Euler constant, $\varepsilon$ denotes any fixed positive number.

Zhao Yuan-e [5] had studied an equation involving the function $S_p(n)$, and obtained some interesting results: let $p$ be a fixed prime, for any positive integer $n$ with $n \leq p$, the equation

$$\sum_{d|n} S_p(d) = 2pn$$

holds if and only if $n$ be a perfect number. If $n$ be an even perfect number, then $n = 2^{r-1}(2^r - 1), r \geq 2$, where $2^r - 1$ is a Mersenne prime.

In this paper, we shall use the elementary methods to study the asymptotic properties of $S_k(n)$, and get a more general asymptotic formula. That is, we shall prove the following conclusion:

**Theorem.** For any fixed positive integer $k > 1$ and any positive integer $n$, we have the asymptotic formula

$$S_k(n) = \alpha(p-1)n + O\left(\frac{p}{\ln p} \ln n\right),$$

where $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of $k$ into prime powers, and $\alpha(p-1) = \max\limits_{1 \leq i \leq r} \{\alpha_i(p_i - 1)\}$.

## §2. Some lemmas

To complete the proof of Theorem, we need the following several lemmas. First for any fixed prime $p$ and positive integer $n$, we let $\alpha(n, p)$ denote the sum of the base $p$ digits of $n$. That is, if $n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$, where $1 \leq a_i \leq p-1, i = 1, 2, \cdots, s$, then $\alpha(n, p) = \sum\limits_{i=1}^{s} \alpha_i$, and for this number theoretic function, we have the following:

**Lemma 1.** For any integer $n \geq 1$, we have the identity

$$\alpha_p(n) \equiv \alpha(n) \equiv \sum_{i=1}^{+\infty} \left[\frac{n}{p^i}\right] = \frac{1}{p-1}(n - \alpha(n, p)),$$

where $[x]$ denotes the greatest integer not exceeding $x$.

**Proof.** (See Lemma 1 of reference [2]).

**Lemma 2.** For any positive integer $n$ with $p \mid n$, we have the estimate

$$\alpha(n,p) \leq \frac{p}{\ln p} \ln n.$$

**Proof.** (See Lemma 2 of reference [2]).

## §3. Proof of the theorem

In this section, we use Lemma 1 and Lemma 2 to complete the proof of Theorem. For any fixed positive integer $k$ and any positive integer $n$, let $S_k(n) = m$, and $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Then from the definition of $S_k(n)$, we know that $k^n \mid m!$ and $k^n \nmid (m-1)!$. So we also get $p_1^{\alpha_1 n} p_2^{\alpha_2 n} \cdots p_r^{\alpha_r n} \mid m!$ and $p_1^{\alpha_1 n} p_2^{\alpha_2 n} \cdots p_r^{\alpha_r n} \nmid (m-1)!$. From the definition of F.Smarandache function $S(n)$ we may immediately get $S_k(n) = m = \max\limits_{1 \leq i \leq r} \{S(p_i^{\alpha_i n})\}$.

For convenient, let

$$m_i = S(p_i^{\alpha_i n}),$$

so we have

$$m = \max_{1 \leq i \leq r} \{m_i\}.$$

Let $m_i = a_{i1} p_i^{\beta_{i1}} + a_{i2} p_i^{\beta_{i2}} + \cdots + a_{is} p_i^{\beta_{is}}$ with $\beta_{is} > \beta_{i(s-1)} > \cdots > \beta_{i1} \geq 0$ under the base $p_i$. From the definition of $S(p_i^{\alpha_i n})$, we know that $p_i^{\alpha_i n} \parallel m_i!$, so that $\beta_{i1} \geq 1$. Note that the factorization of $m_i!$ into prime powers is

$$m_i! = \prod_{q \leq m_i} q^{\alpha_q(m_i)},$$

where $\prod\limits_{q \leq m_i}$ denotes the product over all prime $\leq m_i$, and $\alpha_q(m_i) = \sum\limits_{j=1}^{+\infty} \left[ \frac{m_i}{q^j} \right]$. From Lemma 1 we may immediately get the inequality

$$\alpha_{p_i}(m_i) - \beta_{i1} < \alpha_i n \leq \alpha_{p_i}(m_i),$$

or

$$\frac{1}{p_i - 1}(m_i - \alpha(m_i, p_i)) - \beta_{i1} < \alpha_i n \leq \frac{1}{p_i - 1}(m_i - \alpha(m_i, p_i)),$$

$$\alpha_i(p_i - 1)n + \alpha(m_i, p_i) \leq m_i \leq \alpha_i(p_i - 1)n + \alpha(m_i, p_i) + (p_i - 1)(\beta_{i1} - 1).$$

Combining this inequality and Lemma 2 we obtain the asymptotic formula

$$m_i = \alpha_i(p_i - 1)n + O\left( \frac{p_i}{\ln p_i} \ln m_i \right).$$

From above asymptotic formula we can easily see that $m_i$ can achieve the maxima if $\alpha_i(p_i - 1)$ come to the maxima. So taking $\alpha(p-1) = \max\limits_{1 \leq i \leq r} \{\alpha_i(p_i - 1)\}$, we can obtain

$$m = \alpha(p-1)n + O\left( \frac{p}{\ln p} \ln m \right) = \alpha(p-1)n + O\left( \frac{p}{\ln p} \ln n \right).$$

This completes the proof of Theorem.

# References

[1] F. Smaradache, Only problems, not solutions, Xiquan Publishing House, Chicago, 1993, 41-42.

[2] Zhang Wenpeng and Liu Duansen, On the primitive numbers of power $p$ and its asymptotic property, Smaradache Notions, Book series, **13**(2002), 173-175.

[3] Yi Yuan, On the primitive numbers of power $p$ and its asymptotic property, Scientia Magna, **1**(2005), No.1, 175-177.

[4] Xu Zhefeng, Some arithmetical properties of primitive numbers of power $p$, Scientia Magna, **2**(2006), No.1, 9-12.

[5] Zhao Yuan-e, An equation involving the function $S_p(n)$, Scientia Magna, **2**(2006), No.2, 105-107.

[6] Tom M.Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 22-25**

# On the F.Smarandache LCM function and its mean value

Zhongtian Lv

Basic Courses Department, Xi'an Medical College,
Xi'an, Shaanxi, P.R.China

Received Jan. 9, 2007

**Abstract** For any positive integer $n$, the F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer $k$ such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2,\ \cdots,\ k$. The main purpose of this paper is to use the elementary methods to study the mean value of the F.Smarandache LCM function $SL(n)$, and give a sharper asymptotic formula for it.

**Keywords** F.Smarandache LCM function, mean value, asymptotic formula.

## §1. Introduction and results

For any positive integer $n$, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer $k$ such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2,\ \cdots,\ k$. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $\cdots$. About the elementary properties of $SL(n)$, some authors had studied it, and obtained some interesting results, see reference [3] and [4]. For example, Murthy [3] showed that if $n$ is a prime, then $SL(n) = S(n)$, where $S(n)$ denotes the Smarandache function, i.e., $S(n) = \min\{m :\ n|m!,\ m \in N\}$. Simultaneously, Murthy [3] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [4] completely solved this problem, and proved the following conclusion:

Every positive integer $n$ satisfying (1) can be expressed as

$$n = 12 \quad \texttt{or} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1,\ p_2,\ \cdots,\ p_r,\ p$ are distinct primes, and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1,\ 2,\ \cdots,\ r$.

The main purpose of this paper is to use the elementary methods to study the mean value properties of $SL(n)$, and obtain a sharper asymptotic formula for it. That is, we shall prove the following conclusion:

**Theorem.** Let $k \geq 2$ be a fixed integer. Then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where $c_i$ $(i = 2,\ 3, \cdots, k)$ are computable constants.

From our Theorem we may immediately deduce the following:

**Corollary.** For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

## §2. Proof of the theorems

In this section, we shall prove our theorem directly. In fact for any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of $n$, then from [3] we know that

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_s^{\alpha_s}\}. \tag{2}$$

Now we consider the summation

$$\sum_{n\leq x} SL(n) = \sum_{n\in A} SL(n) + \sum_{n\in B} SL(n), \tag{3}$$

where we have divided the interval $[1,\ x]$ into two sets $A$ and $B$. $A$ denotes the set involving all integers $n \in [1,\ x]$ such that there exists a prime $p$ with $p|n$ and $p > \sqrt{n}$. And $B$ denotes the set involving all integers $n \in [1,\ x]$ with $n \notin A$. From (2) and the definition of $A$ we have

$$\sum_{n\in A} SL(n) = \sum_{\substack{n\leq x \\ p|n,\ \sqrt{n}<p}} SL(n) = \sum_{\substack{pn\leq x \\ n<p}} SL(pn) = \sum_{\substack{pn\leq x \\ n<p}} p = \sum_{n\leq\sqrt{x}} \sum_{n<p\leq\frac{x}{n}} p. \tag{4}$$

By Abel's summation formula (See Theorem 4.2 of [5]) and the Prime Theorem (See Theorem 3.2 of [6]):

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where $a_i$ $(i = 1,\ 2, \cdots, k)$ are constants and $a_1 = 1$.

We have

$$\begin{aligned} \sum_{n<p\leq\frac{x}{n}} p &= \frac{x}{n} \cdot \pi\left(\frac{x}{n}\right) - n \cdot \pi(n) - \int_n^{\frac{x}{n}} \pi(y)dy \\ &= \frac{x^2}{2n^2 \ln x} + \sum_{i=2}^{k} \frac{b_i \cdot x^2 \cdot \ln^i n}{n^2 \cdot \ln^i x} + O\left(\frac{x^2}{n^2 \cdot \ln^{k+1} x}\right), \end{aligned} \tag{5}$$

where we have used the estimate $n \leq \sqrt{x}$, and all $b_i$ are computable constants.

Note that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, and $\sum_{n=1}^{\infty}\frac{\ln^i n}{n^2}$ is convergent for all $i=2,\ 3,\ \cdots,\ k$. From (4) and (5) we have

$$\begin{aligned}\sum_{n\in A}SL(n) &= \sum_{n\le\sqrt{x}}\left(\frac{x^2}{2n^2\ln x}+\sum_{i=2}^{k}\frac{b_i\cdot x^2\cdot\ln^i n}{n^2\cdot\ln^i x}+O\left(\frac{x^2}{n^2\cdot\ln^{k+1}x}\right)\right)\\ &= \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),\end{aligned}\tag{6}$$

where $c_i$ $(i=2,\ 3,\cdots,k)$ are computable constants.

Now we estimate the summation in set $B$. Note that for any positive integer $\alpha$, the series $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{\alpha+1}{\alpha}}}$ is convergent, so from (2) and the definition of $B$ we have

$$\begin{aligned}\sum_{n\in B}SL(n) &= \sum_{\substack{n\le x\\ SL(n)=p,\ p\le\sqrt{n}}}p+\sum_{\substack{n\le x\\ SL(n)=p^\alpha,\ \alpha>1}}p^\alpha\\ &\ll \sum_{\substack{n\le x\\ p|n,\ p\le\sqrt{n}}}p+\sum_{2\le\alpha\le\ln x}\ \sum_{p\le x}\ \sum_{np^\alpha\le x}p^\alpha\\ &\ll \sum_{n\le x}\ \sum_{p\le\min\{n,\ \frac{x}{n}\}}p+\sum_{2\le\alpha\le\ln x}\ \sum_{n\le x}\ \sum_{p\le\left(\frac{x}{n}\right)^{\frac{1}{\alpha}}}p^\alpha\\ &\ll \frac{x^{\frac{3}{2}}}{\ln x}+\frac{x^{\frac{3}{2}}}{\ln x}\cdot\ln x\ll x^{\frac{3}{2}}.\end{aligned}\tag{7}$$

Combining (3), (6) and (7) we may immediately deduce that

$$\sum_{n\le x}SL(n)=\frac{\pi^2}{12}\cdot\frac{x^2}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),$$

where $c_i$ $(i=2,\ 3,\cdots,k)$ are computable constants.

This completes the proof of Theorem.

# References

[1] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] I.Balacenoiu and V.Seleacu, History of the Smarandache function, Smarandache Notions Journal,**10**(1999), 192-201.

[3] A.Murthy, Some notions on least common multiples, Smarandache Notions Journal, **12**(2001), 307-309.

[4] Le Maohua, An equation concerning the Smarandache LCM function, Smarandache Notions Journal, **14**(2004), 186-188.

[5] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[6] Pan Chengdong and Pan Chengbiao, The elementary proof of the prime theorem, Shanghai Science and Technology Press, Shanghai, 1988.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 26-28**

# A conjecture involving the F. Smarandache LCM function

Qibin Wu

Department of Mathematics, Xianyang Normal College
Xianyang, Shaanxi, P.R.China

Received Jan. 15, 2007

**Abstract** For any positive integer $n$, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer $k$ such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2,\ \cdots,\ k$. The main purpose of this paper is to propose a conjecture involving the F.Smarandache LCM function $SL(n)$, and solved it partly.

**Keywords** F.Smarandache LCM function, conjecture, elementary method.

## §1. Introduction and results

For any positive integer $n$, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer $k$ such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2,\ \cdots,\ k$. For example, $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $SL(16) = 16$, $\cdots$. From the definition of $SL(n)$ we can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of $n$ into prime powers, then

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\}.$$

About the other elementary properties of $SL(n)$, many people's had studied it, and obtained a series interesting results, see references [3] and [4]. For example, Murthy [3] porved that if $n$ be a prime, then $SL(n) = S(n)$, where $S(n)$ be the F.Smarandache function. That is, $S(n) = \min\{m :\ n|m!,\ m \in N\}$.

Simultaneously, Murthy [3] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [4] solved this problem completely, and proved the following conclusion: for any positive integer $n$ satisfying (1) can be expressed as

$$n = 12 \quad \texttt{or} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1$, $p_2$, $\cdots$, $p_r$, $p$ are distinct primes and $\alpha_1$, $\alpha_2$, $\cdots$, $\alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

Now we consider the summation:

$$\sum_{d|n} \frac{1}{SL(d)}, \tag{2}$$

where $\sum_{d|n}$ denotes the summation over all positive divisors of $n$. We find that there is no any positive integer $n > 1$ such that (2) is an integer. So in this paper, we propose the following:

**Conjecture.** There is no any positive integer $n \geq 2$ such that (2) is an integer.

We believe that this conjecture is true, even if we can not prove it. The main purpose of this paper is to study this problem, and prove that for some special positive integers $n$, this conjecture is true. That is, we shall prove the following:

**Theorem 1.** Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of $n$ into primes powers (where $p_1 < p_2 < \cdots < p_r$). If $\alpha_1 = 1$, then the conjecture is true.

**Theorem 2.** For any integer $n > 1$, if $SL(n)$ be a prime, then the conjecture is true.

**Theorem 3.** Let $p$ be a prime and $\alpha$ be any positive integer. If $n = p^{\alpha}$, then the conjecture is true.

From Theorem 2 we may immediately deduce the following:

**Corollary.** If $n$ be a square-free number ( $n > 1$, and any prime $p|n \Longrightarrow p^2 \dagger n$), then the conjecture is true.

## §2. Proof of the theorems

In this section, we shall complete the proof of our theorems directly. First we prove Theorem 1. For any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of $n$ into primes powers, then from [3] we know that

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\}. \tag{3}$$

Now if $\alpha_1 = 1$ and $n$ satisfying

$$\sum_{d|n} \frac{1}{SL(d)} = m,$$

is a positive integer. Let $n = p_1 \cdot n_1$, then note that for any $d|n_1$ with $d > 1$, $SL(p_1 \cdot d) = SL(d)$, we have

$$\begin{aligned} m &= \sum_{d|n} \frac{1}{SL(d)} = \sum_{d|n_1} \frac{1}{SL(d)} + \sum_{d|n_1} \frac{1}{SL(p_1 \cdot d)} \\ &= \sum_{d|n_1} \frac{1}{SL(d)} + \sum_{d|n_1} \frac{1}{SL(d)} + \frac{1}{p_1} - 1 = \sum_{d|n_1} \frac{2}{SL(d)} + \frac{1}{p_1} - 1. \end{aligned}$$

or

$$n_1 \cdot m = \sum_{d|n_1} \frac{2n_1}{SL(d)} + \frac{n_1 \cdot (1 - p_1)}{p_1}. \tag{4}$$

It is clear that for any $d|n_1$, $\frac{n_1}{SL(d)}$ and $n_1 \cdot m$ are integers, but $\frac{n_1 \cdot (1-p_1)}{p_1}$ is not an integer. This contradiction with (4). So if $\alpha_1 = 1$, then Theorem 1 is true. This proves Theorem 1.

Now we prove Theorem 2. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of $n$ into primes powers. If $SL(n)$ be a prime, then $SL(n) = p_s$ and $\alpha_s = 1$. This time let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s = n_1 \cdot p_s$. So if (2) is an integer $m$, then note that $SL(p_s \cdot d) = p_s$ for any $d|n_1$, we have

$$\begin{aligned} m &= \sum_{d|n} \frac{1}{SL(d)} = \sum_{d|n_1} \frac{1}{SL(d)} + \sum_{d|n_1} \frac{1}{SL(p_s \cdot d)} \\ &= \sum_{d|n_1} \frac{1}{SL(d)} + \sum_{d|n_1} \frac{1}{p_s} = \sum_{d|n_1} \frac{1}{SL(d)} + \frac{d(n_1)}{p_s}, \end{aligned} \tag{5}$$

where $d(n_1)$ denotes the Dirichlet divisor function of $n_1$. It is clear that for any $d|n_1$, we have $(SL(d), p_s) = 1$. So from (5) we may immediately deduce that

$$p_s \mid d(n_1) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_{s-1} + 1).$$

Without loss of generality, we can assume $p_s \mid \alpha_i + 1$ with $1 \le i \le s-1$. This time we have $\alpha_i + 1 \ge p_s$ or $\alpha_i \ge p_s - 1$. But in this case, we have $p_i^{\alpha_i} \ge p_i^{p_s - 1} \ge (1+1)^{p_s - 1} > p_s$. This contradicts with $SL(n) = p_s$. This proves Theorem 2.

Now we prove Theorem 3. Let $p$ be a prime and $n = p^\alpha$. Then we have

$$\sum_{d|n} \frac{1}{SL(d)} = \sum_{i=0}^{\alpha} \frac{1}{SL(p^i)} = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \frac{1}{p^\alpha} = \frac{1 + p + p^2 + \cdots p^\alpha}{p^\alpha}. \tag{6}$$

Since $\left(p^\alpha, 1 + p + p^2 + \cdots p^\alpha\right) = 1$, so it is not possible that (6) be an integer. This completes the proof of Theorem 3.

## References

[1] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] I.Balacenoiu and V.Seleacu, History of the Smarandache function, Smarandache Notions Journal, **10**(1999), 192-201.

[3] A.Murthy, Some notions on least common multiples, Smarandache Notions Journal, **12**(2001), 307-309.

[4] Le Maohua, An equation concerning the Smarandache LCM function, Smarandache Notions Journal, **14**(2004), 186-188.

[5] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[6] Pan Chengdong and Pan Chengbiao, The elementary proof of the prime theorem, Shanghai Science and Technology Press, Shanghai, 1988.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 29-32**

# On the Smarandache dual function

Shejiao Xue

Department of Mathematics, Weinan Teacher's College
Weinan, Shaanxi, P.R.China

Received Nov. 11, 2006

**Abstract** For any positive integer $n$, the Smarandache dual function $S^*(n)$ is defined as the greatest positive $m$ such that $m!$ divides $n$. The main purpose of this paper is using the elementary method to study the calculating problem of a Dirichlet series involving the Smarandache dual function $S^*(n)$, and give an exact calculating formula for it.

**Keywords** Smarandache dual function, Dirichlet series, exact calculating formula.

## §1. Introduction and results

For any positive integer $n$, the Smarandache dual function $S^*(n)$ is defined as the greatest positive $m$ such that $m!$ divides $n$. That is,

$$S^*(n) = \max\{m : \ m! \mid n \ , \ m \in N\}.$$

The first few values of $S^*(n)$ are $S^*(1) = 1$, $S^*(2) = 2$, $S^*(3) = 1$, $S^*(4) = 2$, $S^*(5) = 1$, $S^*(6) = 3$, $S^*(7) = 1$, $S^*(8) = 2$, $S^*(9) = 1$, $S^*(10) = 2$, $S^*(11) = 1$, $S^*(12) = 3$, $S^*(13) = 1$, $S^*(14) = 2$, $S^*(15) = 1$, $\cdots$. This function was introduced by J.Sandor in [1], where he studied the elementary properties of $S^*(n)$, and obtained a series interesting results. In reference [2], J.Sandor also proposed the following conjecture:

$$S^* \left((2k-1)!(2k+1)!\right) = q - 1,$$

where $k$ is a positive integer, $q$ is the first prime following $2k+1$.

Maohua Le [3] proved that this conjecture is true. The other contents related to the Smarandache dual function can also be found in references [4], [5], [6] and [7]. For example, in reference [6], J.Sandor studied the elementary properties of $S_*(x)$, which called the additive analogue of $S^*(n)$, and proved that

$$S_*(x) \sim \frac{\ln x}{\ln \ln x}, \ x \to \infty,$$

where $S_*(x) = \max\{m : \ m! \leq x, \ m \in N\}$.

In this paper, we use the elementary method to study the calculating problem of the series

$$\sum_{n=1}^{\infty} \frac{S^*(n)}{n^s}, \tag{1}$$

and give an exact calculating formula for (1). At the same time, we also study the mean value properties of $S^*(n)$, and give a sharper mean value formula for $S^*(n)$. That is, we shall prove the following conclusions:

**Theorem 1.** For any real number $s > 1$, we have the identities

$$\sum_{n=1}^{\infty} \frac{(S^*(n))^k}{n^s} = \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{n^k - (n-1)^k}{(n!)^s}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{S^*(n)n^s} = \zeta(s) \cdot \left(1 - \sum_{n=1}^{\infty} \frac{1}{n(n+1)((n+1)!)^s}\right),$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta-function.

Note that $\zeta(2) = \frac{\pi^2}{6}$, $\lim_{s\to 1}(s-1)\zeta(s) = 1$, $\sum_{n=1}^{\infty} \frac{1}{n!} = e - 1$ and $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$, where $\mu(n)$ is the Möbius function. From Theorem 1 we may immediately deduce the following two Corollaries:

**Corollary 1.** For any positive integer $n$, we have the identities

$$\sum_{d|n} \mu(d) S^*\left(\frac{n}{d}\right) = \begin{cases} 1, \text{ if } n = m!, \ m \text{ is any positive integer;} \\ 0, \text{ otherwise.} \end{cases}$$

and

$$\sum_{n=1}^{\infty} \frac{S^*(n)}{n^2} = \frac{\pi^2}{6} \cdot \sum_{n=1}^{\infty} \frac{1}{(n!)^2}.$$

**Corollary 2.** Under the notations of Theorem 1, we have

$$\lim_{s\to 1}(s-1) \cdot \left(\sum_{n=1}^{\infty} \frac{S^*(n)}{n^s}\right) = e - 1,$$

where $e = 2.718281828459\cdots$ is a constant.

It is clear that using Theorem 1 and the Perron's formula (See Theorem 6.5.2 of [7]) we can also give an asymptotic formula for the mean value of $S^*(n)$, but using the elementary method we can deduce the following sharper estimate:

**Theorem 2.** For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} S^*(n) = (e-1)x + O\left(\frac{\ln^2 x}{(\ln\ln x)^2}\right).$$

## §2. Proof of the theorems

In this section, we shall prove our theorems directly. In fact for any positive integer $m$, from the Stirling's formula (See Theorem 3.3.1 of [7]) we know that

$$\ln(m!) = \sum_{k=1}^{m} \ln k = m\ln m - m + O(1). \tag{2}$$

Combining this asymptotic formula and the definition of $S^*(n)$ we can deduce that if $m!|n$, then $m! \le n$ or $\ln(m!) \le \ln n$. So $S^*(n) = m \le \frac{2\ln n}{\ln\ln n}$ and $\frac{S^*(n)}{n^s} \le \frac{2\ln n}{n^s \cdot \ln\ln n}$. Therefore, the Dirichlet series $\sum_{n=1}^{\infty} \frac{S^*(n)}{n^s}$ is absolute convergent, if $s > 1$. Now if $S^*(n) = m$, then $m!|n$. Let $n = m! \cdot n_1$ with $(m+1) \dagger n_1$. So for any real number $s > 1$, from the definition of $S^*(n)$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(S^*(n))^k}{n^s} &= \sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ S^*(n)=m}}^{\infty} \frac{m^k}{n^s} = \sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ (m+1)\dagger n}}^{\infty} \frac{m^k}{(m!)^s \cdot n^s} = \sum_{m=1}^{\infty} \frac{m^k}{(m!)^s} \sum_{\substack{n=1 \\ (m+1)\dagger n}}^{\infty} \frac{1}{n^s} \\
&= \sum_{m=1}^{\infty} \frac{m^k}{(m!)^s} \left( \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(m+1)^s \cdot n^s} \right) \\
&= \left( \sum_{n=1}^{\infty} \frac{1}{n^s} \right) \left( \sum_{m=1}^{\infty} \frac{m^k}{(m!)^s} - \sum_{m=1}^{\infty} \frac{m^k}{((m+1)!)^s} \right) \\
&= \left( \sum_{n=1}^{\infty} \frac{1}{n^s} \right) \left( 1 + \sum_{m=1}^{\infty} \frac{(m+1)^k}{((m+1)!)^s} - \sum_{m=1}^{\infty} \frac{m^k}{((m+1)!)^s} \right) \\
&= \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{n^k - (n-1)^k}{(n!)^s}.
\end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. For any real number $x > 1$, let $k$ be a positive integer such that $k! \le x < (k+1)!$. Then from (2) we can deduce that

$$k = \frac{\ln x}{\ln\ln x} + O\left( \frac{\ln x}{(\ln\ln x)^2} \right).$$

By this estimate and the definition of $S^*(n)$ we have

$$\begin{aligned}
\sum_{n \le x} S^*(n) &= \sum_{m! \le x} \sum_{\substack{n \le x \\ S^*(n)=m}} m = \sum_{\substack{m! \cdot n \le x \\ m+1 \dagger n}} m = \sum_{m! \le x} m \sum_{\substack{n \le \frac{x}{m!} \\ m+1 \dagger n}} 1 \\
&= \sum_{m! \le x} m \left( \sum_{n \le \frac{x}{m!}} 1 - \sum_{n \le \frac{x}{(m+1)!}} 1 \right) = \sum_{m! \le x} m \left( \frac{x \cdot m}{(m+1)!} + O(1) \right) \\
&= x \cdot \sum_{m! \le x} \frac{m^2}{(m+1)!} + O\left( \sum_{m! \le x} m \right) \\
&= x \cdot \sum_{m \le \frac{\ln x}{\ln\ln x}} \frac{m^2}{(m+1)!} + O\left( \frac{\ln x}{(\ln\ln x)^2} \frac{\ln x}{\ln\ln x} \right) + O\left( \frac{\ln^2 x}{(\ln\ln x)^2} \right) \\
&= x \cdot \sum_{m=1}^{\infty} \frac{m^2}{(m+1)!} + O\left( x \cdot \sum_{m! > x} \frac{m^2}{(m+1)!} \right) + O\left( \frac{\ln^2 x}{(\ln\ln x)^2} \right) \\
&= x \cdot \sum_{m=1}^{\infty} \left( \frac{1}{(m-1)!} - \frac{1}{m!} + \frac{1}{(m+1)!} \right) + O\left( \frac{\ln^2 x}{(\ln\ln x)^2} \right) \\
&= (e-1) \cdot x + O\left( \frac{\ln^2 x}{(\ln\ln x)^2} \right),
\end{aligned}$$

where we have used the identity $\sum_{n=0}^{\infty}\frac{1}{n!}=e$. This proves Theorem 2.

Now we prove Corollary 1, note that $\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$, from Theorem 1 we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(n!)^s} &= \left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\right)\cdot\left(\sum_{n=1}^{\infty}\frac{S^*(n)}{n^s}\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu(m)\cdot S^*(n)}{(mn)^s}\\ &= \sum_{n=1}^{\infty}\frac{\sum\limits_{u\cdot v=n}\mu(u)S^*(v)}{n^s}=\sum_{n=1}^{\infty}\frac{\sum\limits_{d|n}\mu(d)S^*\left(\frac{n}{d}\right)}{n^s}. \qquad (3)\end{aligned}$$

Comparing the coefficients of the Dirichlet series in (3), we may immediately get the identity

$$\sum_{d|n}\mu(d)S^*\left(\frac{n}{d}\right)=\begin{cases}1, \text{ if } n=m!,\ m \text{ is any positive integer;}\\ 0, \text{ otherwise.}\end{cases}$$

This completes the proof of the theorems.

# References

[1] J.Sandor, On certain generalizations of the Smarandache function, Notes Numb. Th. Discr. Math., **11**(1999), 45-51.

[2] J.Sandor, On certain generalizations of the Smarandache function, Smarandache Notions Journal, **11**(2000), 202-212.

[3] Maohua Le, A conjecture concerning the Smarandache dual functions, Smarandache Notions Journal, **14**(2004), 153-155.

[4] David Gorski, The Pseudo Smarandache Functions, Smarandache Notions Journal, **12**(2000), 104-108.

[5] J.Sandor, On additive analogues of certain arithmetic function, Smarandache Notions Journal, **14**(2004), 128-132.

[6] J.Sandor, On additive analogues of the function $S$, Smarandache Notions (book series), **13**(2002), 266-270.

[7] Li Jie, On the Smarandache dual function, Scientia Magna, **2**(2006), No.1, 111-113.

[8] Pan Chengdong and Pan Chengbiao, Foundation of Analytic Number Theory, Science Press, Beijing, 1999, 202-205.

[9] Pan Chengdong and Pan Chengbiao, Elementary Number Theory, Beijing University Press, Beijing, 1992.

[10] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 33-38**

# A new arithmetical function and its asymptotic formula

Na Yuan

Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China

**Abstract** In this paper, we introduce a new arithmetical function $S_k(t,n)$ and use the elementary method to study the mean value properties of this function, then give an interesting asymptotic formula for it.

**Keywords** Arithmetical function, mean value, asymptotic formula.

## §1. Introduction

For any positive integer $n$, the famous Smarandache function $S(n)$ is defined by

$$S(n) = \min\{m \in N : n \mid m!\}.$$

For example, $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $\cdots$. This function was introduced by American-Romanian number theorist professor F.Smarandache (see reference [1]). About the arithmetical properties of $S(n)$, many scholars had studied it, and obtained some interesting conclusions (see references [2] and [3]). For example, Li Hailong and Zhao Xiaopeng [2] studied the mean value of the function $S(a_k(n))$ as follows:

$$\sum_{n\leq x} S(a_k(n)) = \frac{\pi^2 x^{1+\frac{1}{k}}}{6(k+1)\ln x} + O\left(\frac{x^{1+\frac{1}{k}}}{\ln^2 x}\right),$$

where $a_k(n)$ denotes the integer part of $k$-th root of $n$.

At the same time, some scholars also studied another function which has close relations with the Smarandache function (see reference [4] and [5]). It is called the Smarandache double factorial function $Sdf(n)$. About the mean value properties of it, Zhu Minhui [4] gave an asymptotic formula:

$$\sum_{n\leq x} Sdf(n) = \frac{7\pi^2}{24}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

In this paper, we will introduce the Smarandache $t$-factorial function $S_k(t,n)$ which denotes the least positive integer $m$ such that $m!_t$ is divisible by $n^k$. That is,

$$S_k(t,n) = \min\{m \in N : n^k | m!_t\},$$

where $m!_t$ denotes

$$m!_t = m*(m-t)*\cdots*(t+i)*i,\ m \equiv i(\mathrm{mod}\ t),\ i = 0,\ 1,\ \cdots,\ t-1.$$

It is clear that this function is a generalization of $S(n)$. In fact, if we take $k = t = 1$, then $S_1(1, n) = S(n)$. The main purpose of this paper is to study the mean value properties of $S_k(t, n)$, and obtain an interesting asymptotic formula for it. That is, we shall prove the following:

**Theorem.** For any real number $x \geq 2$ and any positive integer $t$, we have the asymptotic formula:

$$\sum_{n\leq x} S_k(t,n) = \begin{cases} \frac{[t(5k-3)+3]\pi^2}{24}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right), & \text{if } 2|t; \\ \\ \frac{[t(k-1)+1]\pi^2}{12}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right), & \text{if } 2\dagger t. \end{cases}$$

**Corollary 1.** For any real number $x \geq 2$ and any positive integer $k$, we have the asymptotic formula:

$$\sum_{n\leq x} S_k(1,n) = \sum_{n\leq x} S_k(n) = \frac{k\pi^2}{12}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

**Corollary 2.** For any real number $x \geq 2$, we have the following:

$$\sum_{n\leq x} S_1(2,n) = \sum_{n\leq x} Sdf(n) = \frac{7\pi^2}{24}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

## §2. Some lemmas

To complete the proof of Theorem, we need the following lemmas:

**Lemma 1.** Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the factorization of $n$ into prime powers, where $p_1,\ p_2,\ \cdots,\ p_r$ are distinct primes and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_r$ are positive integers, then we have

$$S_k(t,n) = \max\{S_k(t,p_1^{\alpha_1}),\ S_k(t,p_2^{\alpha_2}),\ \cdots,\ S_k(t,p_r^{\alpha_r})\}.$$

**Proof.** Let $m_i = S_k(t, p_i^{\alpha_i})$ for $i = 1,\ 2,\ \cdots,\ r$. Then we get

$$(p_i^{\alpha_i})^k \mid (m_i)!_t,\ i = 1,\ 2,\ \cdots,\ r.$$

Let $m = \max\{m_1,\ m_2,\ \cdots,\ m_r\}$, we write

$$(m_i)!_t \mid m!_t,\ i = 1,\ 2,\ \cdots,\ r.$$

Thus

$$(p_i^{\alpha_i})^k \mid m!_t,\ i = 1,\ 2,\ \cdots,\ r.$$

Since $p_1,\ p_2,\ \cdots,\ p_r$ are distinct primes, so we have

$$\gcd(p_i^{\alpha_i}, p_j^{\alpha_j}) = 1,\ 1 \leq i < j \leq k.$$

Therefore, we obtain $n^k \mid m!_t$. It implies that

$$S_k(t,n) \leq m.$$

On the other hand, by the definition of $m$, if $S_k(t,n) < m$, then there exists a prime power $p_i^{\alpha_i}(1 \le i \le r)$, such that

$$(p_i^{\alpha_i})^k \mid S_k(t,n)!_t.$$

We get $n^k \mid S_k(t,n)!_t$, but this is a contradiction. This completes the proof of Lemma 1.

**Lemma 2.** For any positive integer $k$, $t$ and $n$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}P(n)$ be the factorization of $n$ into prime powers, and $P(n) = \max\{k, \sqrt{n}\}$, then we have the identity

$$S_k(t,n) = [t(k-1)+1]P(n),$$

where $P(n)$ denotes the greatest prime divisor of $n$.

**Proof.** From the prime powers factorization of $n$, we may immediately get

$$p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r} < \sqrt{n}.$$

Then we have

$$(p_i^{\alpha_i})^k \mid ([t(k-1)+1]P(n))!_t,\ i = 1,\ 2,\ \cdots,\ r.$$

Thus we can easily obtain

$$(p_i^{\alpha_i})^k \mid ([t(k-1)+1]P(n))!_t.$$

But

$$P^k(n)\dagger([t(k-1)+1]P(n)-1)!_t.$$

So, we have

$$S_k(t,n) = [t(k-1)+1]P(n).$$

This completes the proof of Lemma 2.

**Lemma 3.** $\pi(x)$ denotes all the number of prime which is not exceeding $x$, we have

$$\pi(x) = \frac{x}{\ln x} + \frac{c_1 x}{\ln^2 x} + \cdots + \frac{c_m x}{\ln^m x} + O\left(\frac{x}{\ln^{m+1} x}\right),$$

where $c_i$ $(1 \le i \le m)$ are computable constants. **Proof.** see reference [6].

## §3. Proof of the theorem

In this section, we will use above Lemmas to complete the proof of Theorem. First we separate all positive integer $t$ into two cases:

**Case 1.** If $2|t$, then we have

$$\sum_{n\le x} S_k(t,n) = \sum_{\substack{n\le x\\ 2\dagger n}} S_k(t,n) + \sum_{\substack{n\le x\\ 2\mid n}} S_k(t,n). \tag{1}$$

For the first part of (1), we let the sets $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$\mathcal{A} = \{n : n \le x,\ P(n) > \max(k, \sqrt{n})\},\quad \mathcal{B} = \{n : n \le x,\ p \le \max(k, \sqrt{n})\}.$$

**i)** If $k<\sqrt{n}$, then using the Abel's identity (see reference [7]), we may get

$$\begin{aligned}
&\sum_{\substack{n\in\mathcal{A}\\ 2\nmid n}} S_k(t,n)\\
=&\sum_{\substack{n\le x,\ 2\nmid n\\ P(n)>\max(k,\sqrt{n})}} [t(k-1)+1]P(n)\\
=&\sum_{\substack{2l+1\le x\\ P(2l+1)>\sqrt{n}}} [t(k-1)+1]P(2l+1)\\
=&\sum_{2l+1\le\sqrt{x}}\ \sum_{2l+1\le p\le\frac{x}{2l+1}} [t(k-1)+1]p\\
=&\ [t(k-1)+1]\sum_{2l+1\le\sqrt{x}}\ \sum_{2l+1\le p\le x/(2l+1)} p+O\left(\sum_{2l+1\le\sqrt{x}}\ \sum_{\sqrt{2l+1}\le p\le x/(2l+1)}\sqrt{x}\right)\\
=&\ [t(k-1)+1]\sum_{2l+1\le\sqrt{x}}\ \sum_{2l+1\le p\le x/(2l+1)} p+O\left(x^{\frac{3}{2}}\ln x\right) \qquad (2)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{2l+1\le\sqrt{x}}\ \sum_{2l+1\le p\le x/(2l+1)} p\\
=&\sum_{2l+1\le\sqrt{x}}\left(\frac{x}{2l+1}\pi(\frac{x}{2l+1})-(2l+1)\pi(2l+1)-\int_{\sqrt{x}}^{\frac{x}{2l+1}}\pi(s)ds\right)\\
=&\sum_{2l+1\le\sqrt{x}}\frac{x^2}{2(2l+1)^2\ln\frac{x}{2l+1}}-\frac{(2l+1)^2}{2\ln(2l+1)}+O\left(\frac{x^2}{(2l+1)^2\ln^2\frac{x}{2l+1}}\right)\\
&+O\left(\frac{(2l+1)^2}{\ln^2(2l+1)}\right)+O\left(\frac{x^2}{(2l+1)^2\ln^2\frac{x}{2l+1}}-\frac{(2l+1)^2}{\ln^2(2l+1)}\right). \qquad (3)
\end{aligned}$$

Hence, from (3) and Lemma 3, we have

$$\begin{aligned}
&\sum_{2l+1\le\sqrt{x}}\frac{x^2}{(2l+1)^2\ln\frac{x}{2l+1}}=\sum_{0\le l\le(\sqrt{x}-1)/2}\frac{x^2}{(2l+1)^2\ln\frac{x}{2l+1}}\\
=&\sum_{0\le l\le(\ln x-1)/2}\frac{x^2}{(2l+1)^2\ln x}+O\left(\sum_{(\ln x-1)/2\le l\le(\sqrt{x}-1)/2}\frac{x^2\ln(2l+1)}{(2l+1)^2\ln^2 x}\right)\\
=&\ \frac{\pi^2x^2}{8\ln x}+O\left(\frac{x^2}{\ln^2 x}\right). \qquad (4)
\end{aligned}$$

**ii)** If $k>\sqrt{n}$, then we have

$$\begin{aligned}
&\sum_{\substack{n\in\mathcal{A}\\ 2\nmid n}} S_k(t,n)\\
=&\sum_{\substack{n\le x\ 2\nmid n\\ P(n)>k}} [t(k-1)+1]P(n)
\end{aligned}$$

$$
\begin{aligned}
&= [t(k-1)+1] \sum_{\substack{n\le\sqrt{x}\\ 2\dagger n}} \sum_{k\le p\le \frac{x}{n}} p \\
&= [t(k-1)+1] \sum_{\substack{n\le\sqrt{x}\\ 2\dagger n}} \left( \sum_{n<k\le p\le \frac{x}{n}} p - \sum_{p^s<k<p^{s+1}} p \right) \\
&= [t(k-1)+1] \sum_{\substack{n\le\sqrt{x}\\ 2\dagger n}} \left( \sum_{n<k\le p\le \frac{x}{n}} p - \sum_{e^{\frac{\ln k}{s+1}}<k<e^{\frac{\ln k}{s}}} p \right) \\
&= [t(k-1)+1] \sum_{2l+1\le\sqrt{x}} \sum_{2l+1\le p\le x/(2l+1)} p + O\left(x^{\frac{3}{2}}\ln x\right). \\
&= \frac{[t(k-1)+1]\pi^2x^2}{8\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \qquad (5)
\end{aligned}
$$

Therefore

$$
\sum_{\substack{n\in\mathcal{A}\\ 2\dagger n}} S_k(t,n) = \frac{[t(k-1)+1]\pi^2x^2}{8\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \qquad (6)
$$

Similarly, by Eular summation formula (see reference [7]), we can get

$$
\begin{aligned}
&\sum_{\substack{n\in\mathcal{B}\\ 2\dagger n}} S_k(t,n) \ll \sum_{\substack{n\le x\\ 2\dagger n}} \sqrt{n}\ln n \\
= &\int_1^x \sqrt{h}\ln h dh + \int_1^x (h-[h])(\sqrt{h}\ln h)' dh + \sqrt{x}\ln x(x-[x]) \\
\ll &\ x^{\frac{3}{2}}\ln x. \qquad (7)
\end{aligned}
$$

Combining above (6) and (7), we obtain

$$
\sum_{\substack{n\le x\\ 2\dagger n}} S_k(t,n) = \frac{[t(k-1)+1]\pi^2x^2}{8\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \qquad (8)
$$

For the second part of (1), we notice that $n = 2^\alpha n_1$ where $\alpha$, $n_1$ are positive integers and $2\dagger n_1$, let $S_k(1, 2^\alpha n_1) = \min\{m : (2^\alpha n_1)^k | m!\}$, from the definition of $S_k(2^\alpha n_1)$, we have

$$
\sum_{\substack{n\le x\\ 2|n}} S_k(t,n) = \sum_{\substack{2^\alpha n_1\le x\\ 2|n}} S_k(t,2^\alpha n_1) \ll \sum_{\alpha\le \ln x/\ln 2} \ll \sqrt{x}\ln x. \qquad (9)
$$

and

$$
\sum_{\substack{n\le x\\ 2|n}} S_k(t,n) = t\sum_{\substack{n\le x\\ 2|n}} S_k(1,2^\alpha n_1) + O(\sqrt{x}\ln x) = \frac{tk\pi^2x^2}{12\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \qquad (10)
$$

From (9) and (10), we get

$$
\sum_{\substack{n\le x\\ 2|n}} S_k(t,n) = \frac{tk\pi^2x^2}{12\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \qquad (11)
$$

Combining above (8) and (11), we obtain the following asymptotic formula

$$\sum_{n\leq x} S_k(t,n) = \frac{[t(5k-3)+3]\pi^2}{24}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

**Case 2.** If $2\dagger t$, we can also let the sets $\mathcal{A}$ and $\mathcal{B}$ to be as case 1, by the same way, we have

$$\begin{aligned}\sum_{n\leq x} S_k(t,n) &= \sum_{n\in\mathcal{A}} S_k(t,n) + \sum_{n\in\mathcal{B}} S_k(t,n)\\ &= \frac{[t(k-1)+1]\pi^2}{12}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).\end{aligned}$$

This completes the proof of Theorem.

**Remark.** From the definition of $S_k(t,n)$, we can obtain other arithmetical properties of it, for instance:

**Property 1.** If $2|n$ and $n = 2^\alpha n_1$, where $\alpha$, $n_1$ are positive integers with $2\dagger n_1$, we have

$$S_k(t,n) \leq \max\{S_k(t,2^\alpha),\ tS_k(t,n_1)\}.$$

**Property 2.** Let $p$ be a prime and let $\alpha$ be a positive integer, we have $p^k|S_k(t,p^\alpha)$.

**Property 3.** Let $p$ be the least prime divisor of $n$, we have $S_k(t,n) \geq p^k$.

**Property 4.** For any prime $p$ and any integer $\alpha$, we have

$$(tk-t+1)(p-1)\alpha < S_k(t,p^\alpha) < (p-1)[(tk-t+1)(\alpha+1) + \log_p((tk-t+1)\alpha)].$$

# References

[1] F. Smarandache, Only problems, not Solutions, Xiquan Publ. House, Chicago, 1993.

[2] Li Hailong and Zhao Xiaopeng, On the Smarandache function and the $k$-th roots of a positive integer, Research on Smarandache Problems in Number Theory. Hexis, 2004, 119-122.

[3] Mark Farris and Patrick Mitchell, Bounding the Smarandache function, Smarandache Notions Journal, **13**(2002), 37-42.

[4] Zhu Minhui, On the mean value of the Smarandache double factorial function, Scientia Magna, **2**(2006), No. 2, 27-30.

[5] Felice Russo, Five Properties of the Smarandache double factorial function, Smarandache Notions Journal, **13**(2002), 103-105.

[6] Pan Chengdong, Pan Chengbiao, Fundamental of Analytic Number Theory, Shanghai, 1988.

[7] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 39-45**

# Sequences of pyramidal numbers[1]

Arun Muktibodh, Uzma Sheikh, Dolly Juneja, Rahul Barhate and Yogesh Miglani

Mohota Science College
Umred Rd., Nagpur-440009, India.

Received Nov. 12, 2006

**Abstract** Shyam Sunder Gupta [4] has defined Smarandache consecutive and reversed Smarandache sequences of Triangular numbers. Delfim F.M.Torres and Viorica Teca [1] have further investigated these sequences and defined mirror and symmetric Smarandache sequences of Triangular numbers making use of Maple system. One of the authors A.S.Muktibodh [2] working on the same lines has defined and investigated consecutive, reversed, mirror and symmetric Smarandache sequences of pentagonal numbers of dimension 2 using the Maple system. In this paper we have defined and investigated the s-consecutive, s-reversed, s-mirror and s-symmetric sequences of Pyramidal numbers (Triangular numbers of dimension 3.) using Maple 6.

## §1. Introduction

Figurate number is a number which can be represented by a regular geometrical arrangement of equally spaced points. If the arrangement forms a regular polygon the number is called a polygonal number. Different figurate sequences are formed depending upon the dimension we consider. Each dimension gives rise to a system of figurate sequences which are infinite in number.

In this paper we consider a figurate sequence of Triangular numbers of dimension 3, also called as Pyramidal numbers.

The $n$th Pyramidal number $t_n, n \in N$ is defined by:

$$t_n = \frac{n(n+1)(n+2)}{6}$$

We can obtain the first $k$ terms of Pyramidal numbers in Maple as;

```
> t:= n->(1/6)*n*(n+1)*(n+2):
> first := k -> seq (t(n), n=1...20):
> first(20);

  1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, 455, 560, 680,
        816, 969, 1140, 1330, 1540
```

[1]Acknowledgement: UGC, India has supported this work under project No.23- 245/06.

For constructing Smarandache sequence of Pyramidal numbers we use the operation of concatenation on the terms of the above sequence.This operation is defined as ;

```
> conc :=(n,m)-> n*10^length(m)+m:
```

We define Smarandache consecutive sequence $\{scs_n\}$ for Pyramidal numbers recursively as;
$scs_1 = u_1$,
$scs_n = conc(scs_{n-1}, u_n)$
Using Maple We have obtained first 20 terms of Smarandache consecutive sequence of Pyramidal numbers;

```
>conc :=(n,m)-> n*10^length(m)+m:
> scs_n := (u,n)-> if n = 1 then u(1)else conc(scs_n(u,n-1),u(n))fi:
> scs := (u,n)-> seq (scs_n(u,i),i=1...n):
> scs(t,20);

  1, 14, 1410, 141020, 14102035, 1410203556, 141020355684,
        141020355684120, 141020355684120165, 14102035568412016522 0,
        1410203556841201652202 86, 141020355684120165220286364,
        14102035568412016522028636 4455,
        141020355684120165220286364455560,
        1410203556841201652202863644555606 80,
        14102035568412016522028636445556068 0816,
        141020355684120165220286364455560680816969,
        1410203556841201652202863644555606808169691140,
        14102035568412016522028636445555606808169691140 1330,
        141020355684120165220286364455560680816969114013301540
```

Display of the same sequence in the triangular form is;

```
> show := L -> map(i ->print(i),L):
> show([scs(t,20)]);
```

```
                                    1
                                   14
                                 1410
                               141020
                             14102035
                           1410203556
                         141020355684
                       141020355684120
                     141020355684120165
                   14102035568412016522 0
                 1410203556841201652202 86
```

```
                                  14102035568412016522028636
                                 141020355684120165220286364
                                1410203556841201652202863644 55
                               14102035568412016522028636445 5560
                              141020355684120165220286364455 560680
                             1410203556841201652202863644555 60680816
                            14102035568412016522028636445555 06808169691
                           141020355684120165220286364455556 06808169691140
                          1410203556841201652202863644555560 6808169691140133 0
                         14102035568412016522028636445555606 8081696911401330 1
                        141020355684120165220286364455556068 0816969114013301540
```

The reversed Smarandache sequence (rss)associated with a given sequence $\{u_n\}, n \in N$ is defined recursively as:

$$rss_1 = u_1,$$
$$rss_n = conc(u_n, rss_{n-1}).$$

In Maple we use the following program;

```
> rss_n :=(u,n) -> if n=1 then u(1) else conc(u(n),rss_n(u,n-1)) fi:
> rss := (u,n) -> seq(rss_n(u,i),i=1..n):
```

We get the first 20 terms of reversed smarandache sequence of Pyramidal numbers as;

```
> rss(t,20);
```

```
  1, 41, 1041, 201041, 35201041, 5635201041, 845635201041,
        120845635201041, 165120845635201041, 22016512084563520 1041,
        28622016512084563520 1041, 364286220165120845635201041,
        45536428622016512084563520 1041,
        56045536428622016512084563520 1041,
        68056045536428622016512084563520 1041,
        81668056045536428622016512084563520 1041,
        96981668056045536428622016512084563520 1041,
        114096981668056045536428622016512084563520 1041,
        133011409698166805604553642862201651208456352 01041,
        154013301140969816680560455364286220165120845635201041
```

Smarandache Mirror Sequence (sms)is defined as follows:

$$sms_1 = u_1,$$
$$sms_n = conc(conc(u_n, sms_{n-1}), u_n).$$

The following program gives first 20 terms of Smarandache Mirror sequence of Pyramidal numbers.

```
> sms_n := (u,n) -> if n=1 then
> u(1)
```

```
> else
> conc(conc(u(n),sms_n(u,n-1)),u(n))
> fi:
> sms :=(u,n) ->seq(sms_n(u,i), i=1..n):
> sms(t,20);

  1, 414, 1041410, 20104141020, 352010414102035, 5635201041410203556,
        8456352010414102035568 4, 12084563520104141020355684120,
        16512084563520104141020355684120165,
        22016512084563520104141020355684120165220,
        28622016512084563520104141020355684120165220286,
        36428622016512084563520104141020355684120165220286364,
        45536428622016512084563520104141020355684120165220286364455,
        56045536428622016512084563520104141020355684120165220286364\
        4455560, 68056045536428622016512084563520104141020355684120\
        165220286364455560680, 81668056045536428622016512084563520 1\
        04141020355684120165220286364455560680816, 9698166805604553\
        64286220165120845635201041410203556841201652202863644555606\
        80816969, 11409698166805604553642862201651208456352010414 1\
        02035568412016522028636445556068081696911401330, 154013301140969816680560455364\
        28622016512084563520104141020355684120165220286364455560 68\
        08169691140 13301140969816
```

Finally Smarandache Symmetric sequence (sss) is defined as:

$$sss_{2n-1} = conc(bld(scs_{2n-1}), rss_{2n-1}),$$
$$sss_{2n} = conc(scs_{2n}, rss_{2n}), n \in N,$$

where the function "bld" (But Last Digit) is defined in Maple as

```
> bld := n->iquo(n,10):
```

First 20 terms of Smarandache Symmetric sequence are obtained as

```
> bld :=n-> iquo(n,10):
> conc := (n,m)-> n*10^length(m)+m:
> sss_n := (u,n) -> if type(n,odd) then
> conc(bld(scs_n(u,(n+1)/2)),rss_n(u,(n+1)/2))
> else
> conc(scs_n(u,n/2),rss_n(u,n/2))
> fi:
> sss := (u,n) -> seq(sss_n(u,i), i=1..n):
> sss(t,20);
```

```
1, 11, 141, 1441, 1411041, 14101041, 14102201041, 141020201041,
       141020335201041, 1410203535201041, 14102035555635201041,
       14102035565635201041, 141020355688845635201041,
       1410203556848456352010 41, 14102035568412120845635201041,
       14102035568412012084563 5201041,
       14102035568412016165120845635201041,
       14102035568412016516512084563 5201041,
       14102035568412016522220165120845635201041,
       14102035568412016522022016512084563 5201041
```

We find out primes from a large (first 500) terms of various Smarandache sequences defined so far. We have used Maple 6 on Pentium 3 with 128Mb RAM. We first collect the lists of first 500 terms of the consecutive, reversed, mirror and symmetric sequences of Pyramidal numbers;

```
> st :=time(): Lscs500:=[scs(t,500)]: printf("%a seconds",round(time()-st));
15 seconds
> st :=time(): Lrss500:=[rss(t,500)]: printf("%a seconds",round(time()-st));
20 seconds
> st :=time(): Lsms500:=[sms(t,500)]: printf("%a seconds",round(time()-st));
58 seconds
> st :=time(): Lsss500:=[sss(t,500)]: printf("%a seconds",round(time()-st));
12 seconds
```

Further we find the number of digits in the 500th term of each sequence.

```
> length(Lscs500[500]),length(Lrss500[500]);

                              3283, 3283

> length(Lsms500[500]),length(Lsss500[500]);

                              6565, 2846
```

There exist no prime in the first 500 terms of Smarandache consecutive sequence of Pyramidal numbers;

```
> st:= time():select(isprime,Lscs500);

                                   []

> printf("%a minutes",round((time()-st)/60));
9 minutes
```

There is only one prime in the first 500 terms of reversed Smarandache sequence of Pyramidal numbers;

```
> st:= time():
> select(isprime,Lrss500);

                                [41]

> printf("%a minutes",round((time()-st)/60));
119 minutes
```

There is no prime in the first 500 terms of Smarandache mirror sequence;

```
> st:= time():
> select(isprime,Lsms500);
> printf("%a minutes",round((time()-st)/60));

                                []

177 minutes
```

There is only one prime in the first 500 terms of Smarandache symmetric sequence;

```
> st:= time():
> select(isprime,Lsss500);

                                [11]

> printf("%a minutes",round((time()-st)/60));
90 minutes
```

## §2. Open problems

1) How many Pyramidal numbers are there in the first 500 terms of Smarandache consecutive, mirror, symmetric and reverse symmetric sequences of Pyramidal numbers ?

2) What are those numbers ?

## References

[1] Delfim F.M., Viorica Teca, Consective, Reversed, Mirror and Symmetric Smarandache Sequences of Triangular Numbers, Scientia Magna, **1**(2005), No.2, 39-45.

[2] Muktibodh A.S., Smarandache Sequences of Pentagonal Numbers, Scientia Magna, **2**(2006), No.3.

[3] Muktibodh A.S., Figurate Systems, Bull. Marathwada Mathematical Soc., **6**(2005), No.1, 12-20.

[4] Shyam Sunder Gupta, Smarandache Sequence of Triangular Numbers, Smarandache Notions Journal, **14**, 366-368.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 46-53**

# An efficient hybrid genetic algorithm for continuous optimization problems

Rui Zhang and Siyuan Ma

Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China

Received Dec. 3, 2005

**Abstract** Genetic Algorithms are very commonly used as function optimizers, basically due to their search capability. In this paper, we establish an efficient hybrid genetic algorithm (EHGA) by inserting a local search method to enhance the genetic algorithm, so that it could be more robust and statistically sound. The experimental result show that the proposed algorithm can find optimal or close-to-optimal solutions, and it is more efficient than three other existing GAs.

**Keywords** Genetic algorithm, optimization, local search.

## §1. Introduction

Genetic Algorithm (GA) is a general method for searching a complex space and has had many successes in biology, engineering design, and optimization problems[1−3]. However, experiences in using GA have also shown that, because GA does not make optimum use of the available information, the convergence rate is slow and the local search capability is weak, and as result it is hard to find the optimal with higher precision, This deficiency, resulting from GA's weakness in local search can be remedied by using tools to improve the local search is good at fine-tuning but often falls into local optima. The hybrid approach complements the properties of the genetic algorithm and the local search heuristic methods.

In this paper, we design an efficient hybrid genetic algorithm for continuous optimization problems. The local search method used here originates from statistical experimental design[4]. It can be thought of as being analogous to a kind of crossover that generates better offspring by running a small number of parents; hence, it is efficient to find the better solution in a local search space. The proposed algorithm is developed by inserting the local search method after crossover and mutation operations, together with the elitism to guarantee the convergence of the algorithm.

In section 2 we give a preview of genetic algorithm. The efficient hybrid is described in section 3. And then, in section 4, we evaluate the efficiency of the proposed algorithm and compare the performances with three other existing genetic algorithms. We finally conclude the work in section 5.

## §2. A Preview of Genetic Algorithms

Genetic algorithm, developed by Holland in 1975, is known as astochastic search technique that is very useful and efficient in solving complicated optimization problems. It simulates not only the theory of the survival-of-the-fittest developed by Darwin to motivate the good structure but also the theoretical principles of heredity formulated by Mendel to remain the existing structure as well as search much better one. In a natural genetic system, the chromosomes consist of genes. Each gene has a value and position. The combinations of chromosomes form the total genetic prescription for the construction and operation of some organism[2].Similar to the natural systems, the chromosomes and genes are also used in an artificial genetic algorithm, that is, the parameters related to a possible solution to an optimization problem are encoded to form a chromosomes (called an individual) and every character of the chromosomes corresponds to a gene. In order to evaluate the performance of every individual, A collection of the individuals is called a population. The population evolves iteratively in order to improve the quality of its individual until some termination criterion is achieved. During each generation, the individuals in the current population are rated for their evaluation, and then, a new population is created by using three important genetic operators, i.e, selection, crossover, and mutation.

Denote the population of the $t$th generation by $\overrightarrow{X}(t)$. The steps of GA are described as follows[5].

**Step 1** (initialization) Determine the population size $N$, crossover rate $p_c$, mutation rate $p_m$ and the termination criterion; generate $N$ individuals to be the initial population $\overrightarrow{X}(0)$ randomly; set $t := 0$.

**Step 2** (individual evaluation) Calculate or evaluates the fitness of every individual in $\overrightarrow{X}(t)$.

**Step 3** (population evolution).

**3.1. selection** (parents) Select $M/2$ pairs of parents from $\overrightarrow{X}(t)$ by the selection operator;

**3.2. crossover** Execute crossover on the selected parents to product $M$ temporary individuals with probability $p_c$;

**3.3. mutation** Execute mutation on $M$ temporary individuals respectively with probability $p_m$ to form $M$ candidates;

**3.4. selection**(offspring) Select the better $N$ individuals from the above $M$ candidates in terms of the fitness to generate a new population $\overrightarrow{X}(t+1)$.

**Step 4**(termination examination) If the termination criterion is met, output the individual with the largest fitness as the optimal and stop; otherwise, set $t := t + 1$ and go to Step 2.

On the basis of the analysis above, we know that there are four major preparatory steps required to use the genetic algorithm in solving a problem with GA; (ii) the fitness measure, which is the foundation of evaluating and selecting the individuals; (iii) three genetic operators, which are the search mechanisms to form the new population from generation to generation and (iv) some evolutionary parameters, such as the mutation rate $p_m$, the crossover rate $p_c$ (mutation and crossover are done with certain probabilities) and the maximum of iteration $T_{\texttt{max}}$ etc.

Compared with other optimization methods, GA has the following advantages: (i) encoding

feature: GA takes the certain genetic encoding as the object of the operation leading to solving all kinds of complicated optimization problem uniformly; (ii) strong robustness: GA finds the optimal in the way of population search so that it can search the whole solution space more effectively; (iv) probabilistic search: every genetic operation is executed on the stochastic case resulting in the improved ability to skip the local optimal. With these advantages, GA is a popular algorithm for complicated and difficult global optimization problems.

# §3. Proposed Genetic Algorithm

In this section, we are going to explain our proposed genetic algorithm.

The genetic algorithm used in this paper is real-coded. In real coding representation, each chromosome $A = x_1x_2\cdots x_n$ is encoded with the same length and genes as the vector of decision variables $x = (x_1, x_2, \cdots, x_n)^\top$ directly. In this way, the fitness function is defined naturally as the objective function $f(x)$. It has been confirmed that real number encoding has better performance than either binary or gray encoding for continuous optimization problems, because it can not only improve the precision of the solutions but also be convenient for the design of the genetic operators. The details of the proposed algorithm are as follows.

### 3.1. Characteristics of the Genetic Operators

3.1.1 *Crossover operator*

In the process of biological evolution, two homologous chromosomes are intermixed to generate newer chromosome patterns that produce new individuals or species. The evolution operation of simulating this process is called crossover operator. By crossover,one or several children $C(X, Y)$ are generated from a pair of parent (X,Y) that is selected probabilistically from the population independently. Generally, $C(X, Y)$ can be any form of recombination of the individuals $X$ and $Y$. A typical crossover method used in real-coded GA is arithmetic crossover, which is a linear combination of two individuals. Here we use the convex crossover, which is one of typical forms of arithmetic crossover.

Suppose $p_c$ be the crossover rate (usually taking $0.2 \sim 0.3$) and select $p_c \times N$ pairs of parents. Let $(X, Y)$ be any pair of parents where $X = x_1x_2\cdots x_n$ and $Y = y_1y_2\cdots y_n$, then the child $C(X, Y) = c_1c_2\cdots c_n$ generated from $(X, Y)$ by the crossover operator $C$ satisfies

$$c_i = r_i \times x_i + (1 - r_i)y_i,$$

where $r_i$ is a uniformly random number in $[0, 1]$. Denote the collection of $C(X, Y)$ by $\overrightarrow{C}(t)$

3.1.2 *Mutation operator*

Mutation operator, mimicking the gene mutation in natural evolution, is an operation to explore new search space by introducing new search elements.The function of mutation lies in the requirement of the restoration of genetic diversity that may have been lost in a population because of premature convergence. Hence, mutation is widely used in most work of genetic algorithm to avoid premature convergence. To get this object, Gaussian mutation operator $M$ is selected in this paper.

The operation of mutation begins with the probabilistic selection of individuals from the population. At first, select $p_m \times N$ individuals with the mutation rate $p_m$ (usually taking

$0.1 \sim 0.3$) and suppose $X = x_1x_2\cdots x_n$ be any one of them. Then the altered individual $M(X) = m_1m_2\cdots m_n$ satisfies

$$m_i = x_i + r_i,$$

where $r_i$ is a random variable with normal distribution. The expectation of $r_i$ is 0 and the variance of $r_i$ is $\sigma_i(t)$ varies adaptively with the generation $t$ in the following form

$$\sigma_i(t) = scale \times l_i \times (1 - shrink \times t/T_{\max}),$$

where *scale* is the relative proportion between $\sigma_i$ and $l_i$, *shrink* is the dynamically decreasing rate of $\sigma_i$ and $T_{\max}$ is the maximum number of iteration. The collection of $M(X)$ is denoted by $\overrightarrow{M}(t)$.

Gaussian mutation is one of non-uniform mutations that is an adaptive operation in the level of population[6]. Hence, not only is it capable of preventing premature convergence during the early stage of the evolution, but also it possesses more powerful local search capability during the later stage.

3.1.3 *Selection operator*

The selection operator is an evolution operation that decides which in individuals in the current population can be selected and with how much probability they can be selected to be reproduced into next generation. In the selection operation, any individuals is probabilistically selected on the basis of its fitness. Therefore, in general, the better an individual's fitness is more likely it is to be selected. The major role of the selection operator is to direct the course of GA's search so that the algorithm can search the optimal quickly and avoid invalid search. Three basis methods are proportional-based, ranking-based and nonmonotonic-based, respectively[5]. The selection operator used here is the ranking selection.

Assume that $SN$ individuals will be selected from $\overrightarrow{X}(t)$. First sort all the individuals in $\overrightarrow{X}(t)$ into nondecreasing order with respect to their fitness and label them with $1, 2, \cdots, N$. Write the ranked population as $\overrightarrow{X(t)} = \{X_1, X_2, \cdots, X_N\}$ and define the probability that individual $X_i$ is selected by roulette-wheel scheme to form a new population, denoted by $\overrightarrow{S}(t, SN)$. Generally, ranking selection can avoid premature and remain the diversity of the population.

**3.2. Elitism**

In order to enrich the future generations with specific genetic information of the individual with the best fitness from the current generation, that particular individual is preserved in the next generation. This method of preserving the elite individual is called elitism[5]. Suppose that $EN$ is the number of the elitists preserved in each generation. Sort all the individuals in $\overrightarrow{X(t)}$ into decreasing order with respect to their fitness. Define the first $EN$ ones as the elitists in $\overrightarrow{X(t)}$ and denote them by $\overrightarrow{EL}(t) = (X_1^*, X_2^*, \cdots, X_{EN}^*)$.

Elitism is a kind of evolution mechanism that preserves the best individuals searched by GA up to now. It has been shown that elitism is an executive strategy that guarantees the convergence of GA.

**3.3. Local Search Method**

In this subsection, we will introduce the local search method (LSM) that improves the exploitation of the algorithm. It originates from statistical experimental design for evaluating and implementing improvements in products, process and equipment. The fundamental principle is

to improve the quality of a product by minimizing the effect of the causes of variation without eliminating the causes[8].

If we don't consider the realization of the algorithm, the LSM can be considered being analogous to a kind of crossover that generates better offspring by running a small number of parents.Suppose that $p_1$ is the ratio of the parents who participate in the LSM in the current population. We first select $p_1 \times N$ pairs of parents from $\overline{X}'(t) = \overline{X}(t) \bigcup \overline{C}(t) \bigcup \overline{M}(t)$ ,then define a two-level orthogonal array $L_m(2^{m-1})$ with $m$ rows and $m-1$ columns satisfying $m-1 \geq n$ and $n = 2^k$ where $n$ is the length of the chromosome and $k$ is a positive integer number.We take a pair of parent as an example to describe the process of the LSM in detail as follows.

Assume that $X = x_1x_2\cdots x_n$ and $Y = y_1y_2\cdots y_n$ are the selected parent. Therefore, a new individual space $\overline{Z} = \{Z|Z = z_1z_2\cdots z_n, \quad z_i = x_i \quad \text{or} \quad y_i, \quad i = 1, 2, \cdots, n\}$ with $2^n$ individuals is produced. The main idea of LSM is to find a better and more representative offspring $Z^* = z_1^*z_2^*\cdots z_n^*$ from $\overline{Z}$.

Denote the $i^{th}$ row in $L_m(2^{m-1})$ by $(c_{i1}, c_{i2}, \cdots c_{i,m-1})$ where $c_{il} \in \{1, 2\}(l = 1, 2, \cdots m-1)$. Then by crossover the parent $(X, Y)$ together with $(c_{i1}, c_{i2}, \cdots c_{in})$, a new individual $P^i = P_1^iP_2^i\cdots P_n^i$ is generated where $P_l^i = \begin{cases} x_l, & if \quad c_{il} = 1 \\ y_l, & if \quad c_{il} = 2 \end{cases}$ $(l = 1, 2, \cdots n)$, and it is called an experiment. In doing so, we obtain $m$ experiments $P^1, P^2, \cdots P^m$ in terms of $L_m(2^{m-1})$.

Let $S_i^1 = \{k|c_{ki} = 1, k \in \{1, 2, \cdots m\}\}$ and $S_i^2 = \{k|c_{ki} = 2, k \in \{1, 2, \cdots m\}\}$ represent the set of the rows in which the element equals 1 or 2, respectively, in the $i^{th}$ column. Them for the $i^{th}$ column, we can obtain a pair of data $(Ef_i^1, Ef_2^2)$, by computing $Ef_i^1 = \sum_{j \in s_i^1} \eta_j$ and $Ef_i^2 = \sum_{j \in s_i^2} \eta_j$ $(i = 1, 2, \cdots n)$ where $\eta_j = 1/f^2(P^j)$ $(j = 1, 2, \cdots m)$. Through the comparison of the relation between $Ef_i^1$ and $Ef_i^2$, the most appropriate level of every factor (namely, every gene of a chromosome) is obtained, and then the better offspring $Z^* = z_1^*z_2^*\cdots z_n^*$ is generated with $z_i^* = \begin{cases} x_l, & if \quad Ef_i^1 \geq Ef_i^2 \\ y_l, & if \quad Ef_i^1 < Ef_i^2 \end{cases}$, $i = 1, 2, \cdots n$. It is obvious that $Z^*$ is an offspring reproduced by the parent $X$ and $Y$. So according to the procedure mentioned above, $P_l \times N$ pairs of parents can generate $P_l \times N$ offspring. Denote the collection of offspring by $\overrightarrow{L}(t)$.

On the basis of the mechanism explained in the earlier subsections, the flow chart of the final algorithm is shown in Fig.1.

## §4. Numerical experiments and results

In this section, we are going to have a discussion on the performances among the proposed genetic algorithm (EHGA), the standard genetic algorithm (SGA), the accelerating genetic algorithms (AGA)[9], and the multi-parent crossover genetic algorithms (MGA)[10]. Each of the above algorithms was executed under the following evolutionary environments: the population size is 1500 for EHGA ,SGA, MGA and 500 for AGA; the crossover rate is 0.8 for SGA, MGA and 0.6 for EHGA; the mutation rate is 0.2; the maximum iteration is 105; In EHGA, $p_l$ is 0.1; In MGA, the migration rate is 0.2 and the iteration step between two adjacent migrations is

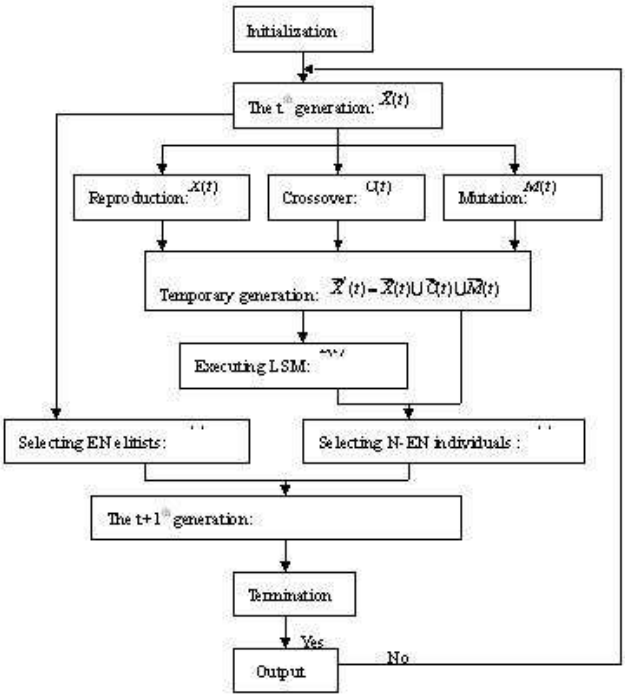



**Fig. 1** Flow chart of the EHGA proposed in this paper

20; In AGA, the number of acceleration is 15 and performs 30 runs at each acceleration. All of these algorithms were tested on the following benchmark functions:

$f_1(x) = \dot{\mathbf{a}}_{i=1}^{30} x_i^2 - \cos(18x_i)$

$f_2(x) = 7 - 20\exp(-0.2\sqrt{\frac{1}{30}\dot{\mathbf{a}}_{i=1}^{30} x_i^2}) - \exp(\frac{1}{30}\dot{\mathbf{a}}_{i=1}^{30}\cos(2px_i))20 + e$

$f_3(x) = \frac{1}{30}\dot{\mathbf{a}}_{i=1}^{30} x_i^4 - 16x_i^2 + 5x_i$

$f_4(x) = \max\{|x_i|, i = 1, \ldots, 30\}$

$f_5(x) = \dot{\mathbf{a}}_{i=1}^{30} x_i^2 + |\mathbf{O}_{i=1}^{30} x_i|$

Table 1 shows the performance of our proposed algorithm comparing with SGA, MGA, and AGA. The computational results indicate that EHGA can, in general, have better quality in the solutions. Fig.2 shows the convergence history of the proposed algorithm for the five test functions. Simulation results show this algorithm is simple in coding, fast in convergence and effective to find the optimal.

Table 1 Comparison among the four GAs

| | EHGA | SGA | AGA | MGA | Global Minimum |
|---|---|---|---|---|---|
| $f_1(x)$ | -29.5156 | -28.2978 | -28.2978 | -29.4158 | -30 |
| $f_2(x)$ | 6.07E-06 | 0.032581 | 0.001733 | 0.046045 | 0 |
| $f_3(x)$ | -78.332 | -68.896 | -68.907 | -70.063 | -78.33236 |
| $f_4(x)$ | 0.059371 | 0.13432 | 0.13226 | 0.13515 | 0 |
| $f_5(x)$ | 5.48E-10 | 0.000103 | 2.38E-07 | 0.003018 | 0 |

## 5. Conclusions

In this paper, an Efficient Hybrid Genetic Algorithm (EHGA) for continuous optimization problems is proposed. The resulting algorithm incorporates a local search method, which can exploit the optimum offspring from a small number of parents, into the traditional genetic algorithm, which contains ranking selection, arithmetic crossover and adaptive Gaussian mutation. Meanwhile, the elitism is adopted to guarantee the convergence of the algorithm. The proposed algorithm is examined by solving 5 benchmark problems with 30 dimensions. The simulation results show that the EHGA can reach the optimal or the close-ti-optimal solutions efficiently, and

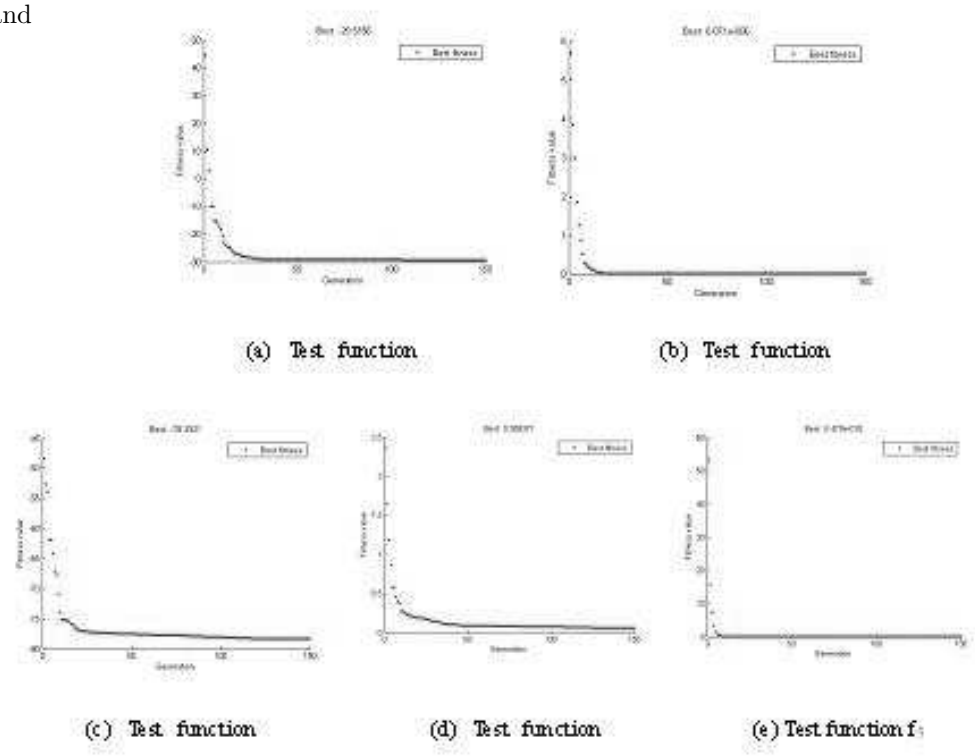



**Fig. 2** Convergence history for the test Functions

it can obtain better results than SGA, MGA, and AGA. However, there are still some difficulties in solving continuous optimization problems with constraints by the proposed EHGA. The main difficulties come from the efficient modelling of selecting the initial population from the search space, and the way of establishing an appropriate penalty function from the original problem. All of these difficulties are under our ongoing research.

## References

[1] J. H. Holland, Adaptation in Natural and Artificial Systems, Ann Arbor, MI, Univ.

Michigan Press, 1975.

[2] D. E. Goldberg, Genetic Algorithms in Search, Optimization and Machine Learning, New York, Addison-Wesley, 1989.

[3] M. Gen and R. Cheng, Genetic Algorithm and Engineering Optimization, John Wiley and Sons, New York, 2000.

[4] D. C. Nontgomery, Design and Analysis of Experiments, New York: Wiley, 1991.

[5] Z. B. Xu, Computational Intelligence-Simulated Evolutionary Computation, Beijing, Higher Education Press, 2004.

[6] Z. Michalewicz, Genetic Algorithm + Data Structure = Evolutionary Programs (3*rd Ed.*), New York, 2000.

[7] J. T. Tsa, T.K.Liu and J.H. Chou, Hybird Taguchi-genetic algorithm for global numerical optimization, IEEE Trans, On Evolutionary Computation, **8**(2004), 365-377.

[8] J. L. Jin, X. H. Yang and J.Ding, An improved simple genetic algorithm-accelerating genetic algorithm, Theory and Practice of System Engineering, **4**(2001), 8-13.

[9] J. Z. Li, L. S. Kang and N.Fang, Multi-parent crossover and its application in continuous optimization, Computer Engineering, **16**(2004), 33-35.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 54-80**

# An introduction to Smarandache multi-spaces and mathematical combinatorics[1]

Linfan Mao
Chinese Academy of Mathematics and System Science
Beijing 100080, P.R.China
E-mail: maolinfan@163.com

Received Jan. 28, 2007

**Abstract** These Smarandache spaces are right theories for objectives by logic. However, the mathematical combinatorics is a combinatorial theory for branches in classical mathematics motivated by a combinatorial speculation. Both of them are unifying theories for sciences and contribute more and more to mathematics in the 21st century. In this paper, I introduce these two subjects and mainly concentrate on myself research works on mathematical combinatorics finished in past three years, such as those of map geometries, pseudo-manifolds of dimensional $n$, topological or differential structures on smoothly combinatorial manifolds. All of those materials have established the pseudo-manifold geometry and combinatorially Finsler geometry or Riemannian geometry. Other works for applications of Smarandache multi-spaces to algebra and theoretical physics are also partially included in this paper.

**Keywords** Smarandache multi-space, mathematical combinatorics, Smarandache $n$-manifold, map geometry, topological and differential structures, geometrical inclusions.

## §1. Introduction

Today, we have known two heartening mathematical theories for sciences. One of them is the Smarandache multi-space theory, came into being by purely logic ([22] – [23]). Another is the mathematical combinatorics motivated by a combinatorial speculation for branches in classical mathematics([7], [16]). The former is more like a philosophical notion. However, the later can be enforced in practice, which opened a new way for mathematics in the 21st century, namely generalizing classical mathematics by its combinatorialization.

Then what is a Smarandache multi-space? Let us begin from a famous proverb. See Fig.1.1. In this proverb, six blind men were asked to determine what an elephant looked like by feeling different parts of the elephant's body.

[1] Reported at The Third International Conference on Number Theory and Smarandache's Problems of China, Mar. 23-25，2007, Xi'an, P.R.China

**Fig.**1.1

The man touched the elephant's leg, tail, trunk, ear, belly or tusk claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, respectively. They entered into an endless argument. Each of them insisted his view right. All of you are right!A wise man explains to them: Why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said.

Certainly, Smarandache multi-spaces are related with the natural space. For this space, a view of the sky by eyes of a man stand on the earth is shown in Fig.1.2. The bioelectric structure of human's eyes decides that he or she can not see too far, or too tiny thing without the help of precision instruments. The picture shown in Fig.1.3 was made by the Hubble telescope in 1995.

**Fig.**1.2

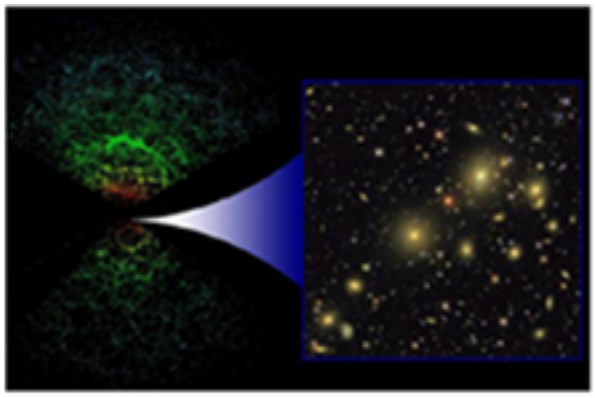


**Fig.**1.3

Physicists are usually to write $(t, x_1, x_2, x_3)$ in $\mathbf{R}^4$ to represent an *event*. For two events $A_1 = (t_1, x_1, x_2, x_3)$ and $A_2 = (t_2, y_1, y_2, y_3)$, their *spacetime interval* $\triangle s$ is defined by

$$\triangle^2 s = -c^2 \triangle t^2 + \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2},$$

where $c$ is the speed of light in the vacuum. The Einstein's general relativity states that all laws of physics take the same form in any reference system and his equivalence principle says that there are no difference for physical effects of the inertial force and the gravitation in a field small enough.

Combining his two principles, Einstein got his gravitational equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu},$$

where

$R_{\mu\nu} = R_{\nu\mu} = R^{\alpha}_{\mu\alpha\nu}$ and $R = g^{\nu\mu}R_{\nu\mu}$, $R^{\alpha}_{\mu i\nu} = \frac{\partial\Gamma^{i}_{\mu i}}{\partial x^{\nu}} - \frac{\partial\Gamma^{i}_{\mu\nu}}{\partial x^{i}} + \Gamma^{\alpha}_{\mu i}\Gamma^{i}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{i}_{\alpha i}$, $\Gamma^{g}_{mn} = \frac{1}{2}g^{pq}(\frac{\partial g_{mp}}{\partial u^{n}} + \frac{\partial g_{np}}{\partial u^{m}} - \frac{\partial g_{mn}}{\partial u^{p}})$.

Applying the Einstein's equation of gravitational field and the cosmological principle, namely there are no difference at different points and different orientations at a point of a cosmos on the metric $10^4 l.y.$ with the Robertson-Walker metric

$$ds^2 = -c^2dt^2 + a^2(t)[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)].$$

Friedmann got a standard model of the universe which classifies universes into three types: static, contracting and expanding. This model also brought about the birth of the Big Bang model in thirties of the 20th century. The following diagram describes the developing process of our cosmos in different periods after the Big Bang.

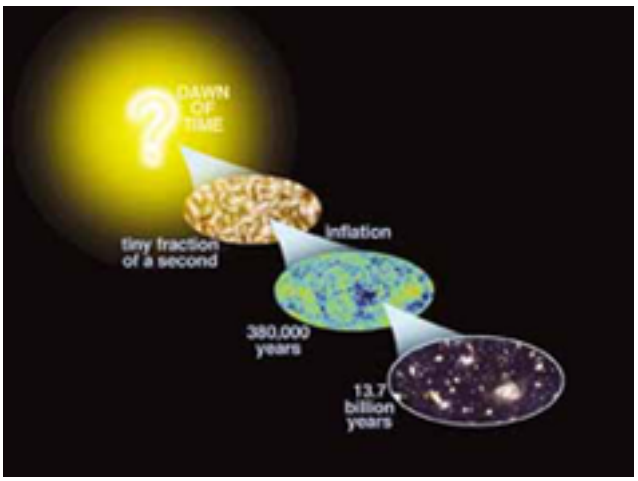



**Fig.**1.4

Today, more and more evidences indicate that our universe is in accelerating expansion. In 1934, R.Tolman first showed that blackbody radiation in an expanding universe cools but retains its thermal distribution and remains a blackbody. G.Gamow, R.Alpher and R.Herman predicted that a Big Bang universe will have a blackbody cosmic microwave background with temperature about 5K in 1948. Afterward, A.Penzias and R.Wilson discovered the 3K cosmic microwave background (CMB) radiation in 1965, which made the two physicists finally won the Noble Prize of physics in 1978. G.F.Smoot and J.C.Mather also won the Noble Prize of physics for their discovery of the blackbody form and anisotry of the cosmic microwave background radiation in 2006. In Fig.1.5, the CMB timeline and a drawing by a artificial satellite WMAP in 2003 are shown.

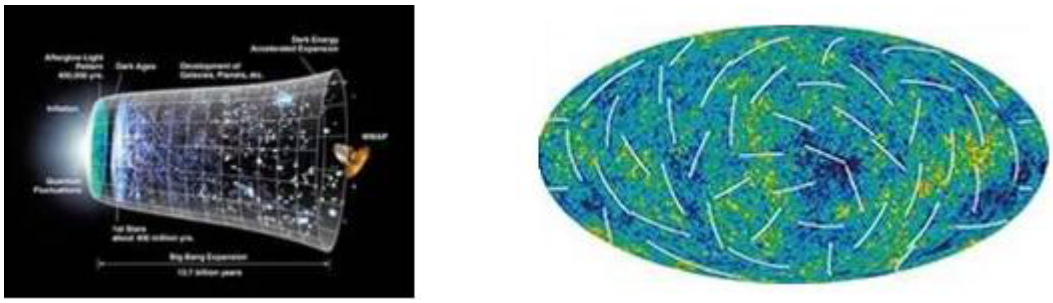


**Fig.**1.5

We have known that all matters are made of atoms and sub-atomic particles, held together by four fundamental forces, i.e., gravity, electromagnetism, strong nuclear force and weak force. They are partially explained by Quantum Theory (electromagnetism, strong nuclear force and weak force) and Relativity Theory(gravity). The Einstein' s unifying theory of fields wishs to describe the four fundamental forces, i.e., combine Quantum Theory and Relativity Theory. His target was nearly realized in 80s in last century, namely the establishing of string/M-theory.

There are five already known string theories, i.e., $E_8 \times E_8$ heterotic string, $SO(32)$ heterotic string, $SO(32)$ Type I string, Type $IIA$ and Type $IIB$, each of them is an extreme theory of M-theory such as those shown in Fig.1.6.

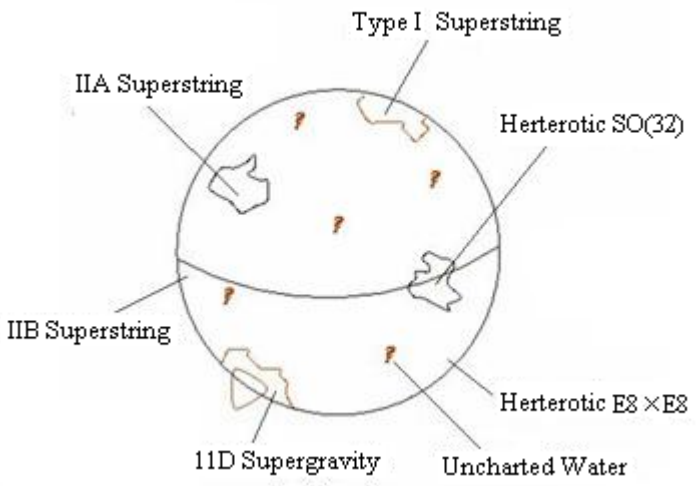



**Fig.**1.6

Then what is the right theory for the universe? A right theory for the universe $\Sigma$ should be

$$\begin{aligned}\Sigma \quad = \quad & \{E_8 \times E_8 \text{ heterotic string}\} \bigcup \{SO(32) \text{ heterotic string}\} \\ & \bigcup \{SO(32) \text{ type I string}\} \bigcup \{\text{type IIA string}\} \\ & \bigcup \{\text{type IIB string}\} \bigcup A \bigcup \cdots \bigcup B \cdots \bigcup C,\end{aligned}$$

where $A, \cdots, B, \cdots, C$ denote some unknown theories for the universe $\Sigma$.

Generally, what is a right theory for an objective $\Delta$? We all know that the foundation of

science is the measures and metrics. Different characteristic $A_i$ by different metric describes the different side $\Delta_i$ of $\Delta$. Therefore, a right theory for $\Delta$ should be

$$\Delta = \bigcup_{i\geq 1} \Delta_i = \bigcup_{i\geq 1} A_i.$$

Now Smarandache multi-spaces are formally defined in the next, which convinces us that Smarandache multi-spaces are nothing but mathematics for right theories of objectives.

**Definition** 1.1.([9],[22]) A Smarandache multi-space is a union of $n$ different spaces equipped with some different structures for an integer $n \geq 2$.

For example, let $n$ be an integer, $Z_1 = (\{0, 1, 2, \cdots, n-1\}, +)$ an additive group ( mod n) and $P = (0, 1, 2, \cdots, n-1)$ a permutation. For any integer $i, 0 \leq i \leq n-1$, define

$$Z_{i+1} = P^i(Z_1),$$

such that $P^i(k) +_i P^i(l) = P^i(m)$ in $Z_{i+1}$ if $k + l = m$ in $Z_1$, where $+_i$ denotes the binary operation $+_i : (P^i(k), P^i(l)) \to P^i(m)$. Then we know that $\bigcup_{i=1}^{n} Z_i$ is a Smarandache multi-space.

The mathematical combinatorics is a combinatorial theory for classical mathematics established by the following conjecture on mathematical sciences.

**Conjecture** 1.1.([7], [16]) Every mathematical science can be reconstructed from or made by combinatorization.

This conjecture means that

$(i)$ One can selects finite combinatorial rulers to reconstruct or make generalization for classical mathematics and

$(ii)$ One can combine different branches into a new theory and this process ended until it has been done for all mathematical sciences.

Applications of the mathematical combinatorics to geometry, algebra and physics can be found in these references [9] – [17]. For terminologies and notations not defined in this paper, we follow [1], [4] for differential geometry and [21], [24] for topology.

## §2. Smaradache Geometries

### 2.1. Geometrical multi-space

A multi-metric space is defined in the following.

**Definition** 2.1 A multi-metric space is a union $\bigcup_{1}^{m} M_i$ such that each $M_i$ is a space with a metric $\rho_i$ for any integer $i$, $1 \leq i \leq m$.

### 2.2. Smarandache geometries

The axiom system of the Euclid geometry consists following five axioms.

(A1) There is a straight line between any two points.

(A2) A finite straight line can produce a infinite straight line continuously.

(A3) Any point and a distance can describe a circle.

(A4) All right angles are equal to one another.

(A5) If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The axiom (A5) can be also replaced by:

(A5') Given a line and a point exterior this line, there is one line parallel to this line.

The Lobachevshy-Bolyai-Gauss geometry, also called hyperbolic geometry is a geometry with axioms $(A1)-(A4)$ and the following axiom $(L5)$:

(L5) There are infinitely many line parallels to a given line passing through an exterior point.

The Riemann geometry, also called elliptic geometry is a geometry with axioms $(A1)-(A4)$ and the following axiom $(R5)$:

(R5) There is no parallel to a given line passing through an exterior point.

These two geometries are mixed non-Euclid geometry. F.Smarandache asked the following question in 1969 for new mixed non-euclid geometries.

**Question** 2.1. Are there other geometries by denying axioms in Euclid geometry not like the hyperbolic or Riemann geometry?

He also specified his question to the following concrete question.

**Question** 2.2. Are there paradoxist geometry, non-geometry, counter-projective geometry and anti-geometry defined by definitions follows?

2.2.1. **Paradoxist geometry**

In this geometry, its axioms are $(A1)-(A4)$ and with one of the following as the axiom $(P5)$.

$(i)$ There are at least a straight line and a point exterior to it in this space for which any line that passes through the point intersect the initial line.

$(ii)$ There are at least a straight line and a point exterior to it in this space for which only one line passes through the point and does not intersect the initial line.

$(iii)$ There are at least a straight line and a point exterior to it in this space for which only a finite number of lines $l_1, l_2, \cdots, l_k, k \geq 2$ pass through the point and do not intersect the initial line.

$(iv)$ There are at least a straight line and a point exterior to it in this space for which an infinite number of lines pass through the point (but not all of them) and do not intersect the initial line.

$(v)$ There are at least a straight line and a point exterior to it in this space for which any line that passes through the point and does not intersect the initial line.

2.2.2. **Non-Geometry**

The non-geometry is a geometry by denial some axioms of $(A1)-(A5)$ such as follows.

$(A1^-)$ It is not always possible to draw a line from an arbitrary point to another arbitrary point.

$(A2^-)$ It is not always possible to extend by continuity a finite line to an infinite line.

$(A3^-)$ It is not always possible to draw a circle from an arbitrary point and of an arbitrary interval.

$(A4^-)$ Not all the right angles are congruent.

$(A5^-)$ If a line, cutting two other lines, forms the interior angles of the same side of it strictly less than two right angle, then not always the two lines extended towards infinite cut each other in the side where the angles are strictly less than two right angle.

2.2.3. **Counter-Projective geometry**

Denoted by $P$ the point set, $L$ the line set and $R$ a relation included in $P \times L$. A counter-projective geometry is a geometry with counter-axioms following.

$(C1)$ There exist: either at least two lines, or no line, that contains two given distinct points.

$(C2)$ Let $p_1, p_2, p_3$ be three non-collinear points, and $q_1, q_2$ two distinct points. Suppose that $\{p_1.q_1, p_3\}$ and $\{p_2, q_2, p_3\}$ are collinear triples. Then the line containing $p_1, p_2$ and the line containing $q_1, q_2$ do not intersect.

$(C3)$ Every line contains at most two distinct points.

2.2.4. **Anti-Geometry**

A geometry by denial some axioms of the Hilbert's 21 axioms of Euclidean geometry.

**Definition** 2.2.([6]) An axiom is said Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(1969).

For example, let us consider an Euclidean plane $\mathbf{R}^2$ and three non-collinear points $A, B$ and $C$. Define $s$-points as all usual Euclidean points on $\mathbf{R}^2$ and $s$-lines as any Euclidean line that passes through one and only one of points $A, B$ and $C$. This geometry then is a Smarandache geometry because two axioms are Smarandachely denied comparing with an Euclid geometry.

$(i)$ The axiom (A5) that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: one parallel and no parallel. Let $L$ be an $s$-line passing through $C$ and not parallel to $AB$ in the Euclidean sense. Notice that through any $s$-point collinear with $A$ or $B$ there is one $s$-line parallel to $L$ and through any other $s$-point there are no $s$-lines parallel to $L$ such as those shown in Fig.2.1$(a)$.

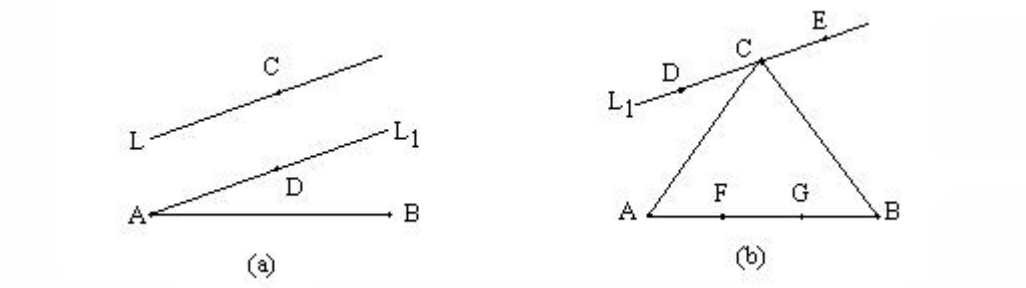



**Fig.**2.1

$(ii)$ The axiom that through any two distinct points there exists one line passing through them is now replaced by; one $s$-line and no $s$-line. Notice that through any two distinct $s$-points $D, E$ collinear with one of $A, B$ and $C$, there is one $s$-line passing through them and through any two distinct $s$-points $F, G$ lying on $AB$ or non-collinear with one of $A, B$ and $C$, there is no $s$-line passing through them such as those shown in Fig.2.1$(b)$.

Iseri constructed $s$-manifolds for dimensional 2 Smarandache manifolds in [5] as follows.

An $s$-manifold is any collection of these equilateral triangular disks $T_i$, $1 \leq i \leq n$ satisfying conditions following:

$(i)$ Each edge $e$ is the identification of at most two edges $e_i, e_j$ in two distinct triangular disks $T_i, T_j, 1 \leq i \leq n$ and $i \neq j$;

$(ii)$ Each vertex $v$ is the identification of one vertex in each of five, six or seven distinct triangular disks, called elliptic, euclidean or hyperbolic point.

These vertices are classified by the number of the disks around them. A vertex around five, six or seven triangular disks is called respective an elliptic vertex, an Euclid vertex or a hyperbolic vertex, which can be realized in $\mathbf{R}^3$ such as shown in Fig.2.2 for an elliptic point and Fig.2.3 for a hyperbolic point.

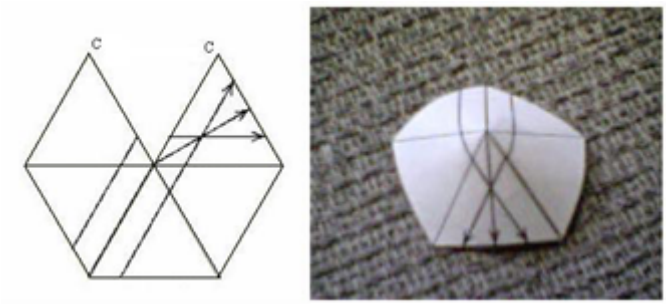



**Fig.**2.2

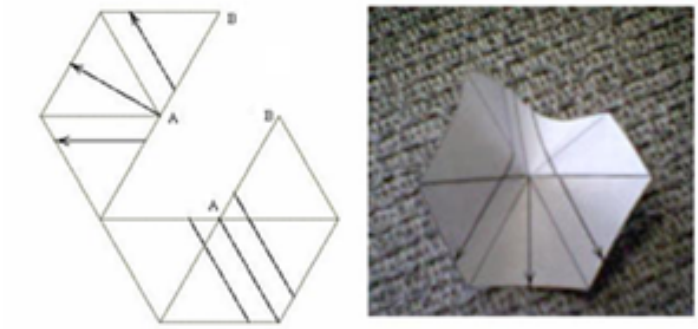



**Fig.**2.3

Iseri proved in [5] that there are Smarandache geometries, particularly, paradoxist geometries, non-geometries, counter-projective geometries and anti-geometries in s-manifolds.

Now let $\Delta_i, 1 \leq i \leq 7$ denote those of closed $s$-manifolds with vertex valency $5, 6, 7$, 5 or 6, 5 or 7, 6 or 7, 5 or 6 or 7, respectively. Then a classification for closed $s$-manifolds was obtained in [7].

**Theorem** 2.1.([7]) $|\Delta_i| = +\infty$ for $i = 2, 3, 4, 6, 7$ and $|\Delta_1| = 2, |\Delta_5| \geq 2$.

2.3. **Smarandache manifolds**

For any integer $n$, $n \geq 1$, an $n$-manifold is a Hausdorff space $M^n$, i.e., a space that satisfies the $T_2$ separation axiom, such that for any $p \in M^n$, there is an open neighborhood $U_p$, $p \in U_p$ a subset of $M^n$ and a homeomorphism $\varphi_p : U_p \to \mathbf{R}^n$ or $\mathbf{C^n}$, respectively.

A Smarandache manifold is an $n$-dimensional manifold that support a Smarandache geometry.

**Question** 2.3. Can we construct Smarandache $n$-manifolds for any integer $n \geq 2$?

# §3. Constructing Smarandache 2-manifolds

3.1. **Maps geometries**

Closed $s$-manifolds in Iseri's model is essentially Smarandache 2-manifolds, special triangulations of spheres with vertex valency $5, 6$ or $7$. A generalization of his idea induced a general construction for Smarandache 2-manifolds, namely map geometries on 2-manifolds.

Let us introduce some terminologies in graph theory first. A graph $G$ is an ordered 3-tuple $(V, E; I)$, where $V, E$ are finite sets, $V \neq \emptyset$ and $I : E \to V \times V$. Call $V$ the vertex set and $E$ the edge set of $G$, denoted by $V(G)$ and $E(G)$, respectively. A graph can be represented by a diagram on the plane, in which vertices are elements in $V$ and two vertices $u, v$ is connected by an edge $e$ if and only if there is a $\varsigma \in I$ enabling $\varsigma(e) = (u, v)$.

The classification theorem for 2-dimensional manifolds in topology says that each 2-manifold is homomorphic to the sphere $P_0$, or to a 2-manifold $P_p$ by adding $p$ handles on $P_0$, or to a 2-manifold $N_q$ by adding $q$ crosscaps on $P_0$. By definition, the former is said an orientable 2-manifold of genus $p$ and the later a non-orientable 2-manifold of genus $q$. This classification for 2-dimensional manifolds can be also described by polygon representations of 2-manifolds with even sides stated following again.

Any compact 2-manifold is homeomorphic to one of the following standard 2-manifolds:

$(P_0)$ the sphere: $aa^{-1}$;

$(P_n)$ the connected sum of $n, n \geq 1$ tori:

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_nb_na_n^{-1}b_n^{-1};$$

$(Q_n)$ the connected sum of $n, n \geq 1$ projective planes:

$$a_1a_1a_2a_2\cdots a_na_n.$$

A combinatorial map $M$ is a connected topological graph cellularly embedded in a 2-manifold $M^2$. For example, the graph $K^4$ on the Klein bottle with one face length 4 and another 8 is shown in Fig.3.1.

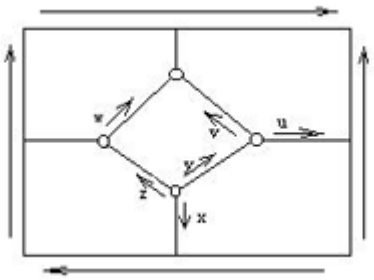


**Fig.**3.1

**Definition** 3.7. For a combinatorial map $M$ with each vertex valency$\geq 3$, endow each vertex $u, u \in V(M)$ a real number $\mu(u), 0 < \mu(u) < \frac{4\pi}{\rho_M(u)}$. Call $(M, \mu)$ a map geometry without boundary, $\mu : V(M) \to R$ an angle function on $M$.

As an example, Fig.3.2 presents a map geometry without boundary on a map $K^4$ on the plane.

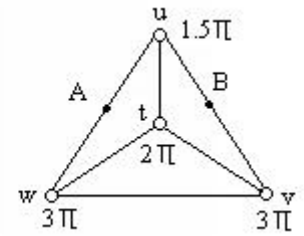



**Fig.**3.2

In this map geometry, lines behaviors are shown in Fig.3.3.

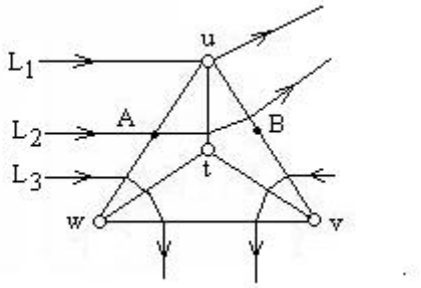



**Fig.**3.3

**Definition** 3.8. For a map geometry $(M, \mu)$ without boundary and faces $f_1, f_2, \cdots, f_l \in F(M), 1 \leq l \leq \phi(M) - 1$, if $S(M) \setminus \{f_1, f_2, \cdots, f_l\}$ is connected, then call $(M, \mu)^{-l} = (S(M) \setminus \{f_1, f_2, \cdots, f_l\}, \mu)$ a map geometry with boundary $f_1, f_2, \cdots, f_l$, where $S(M)$ denotes the locally orientable 2-manifold underlying $M$.

An example for map geometries with boundary is presented in Fig.3.4

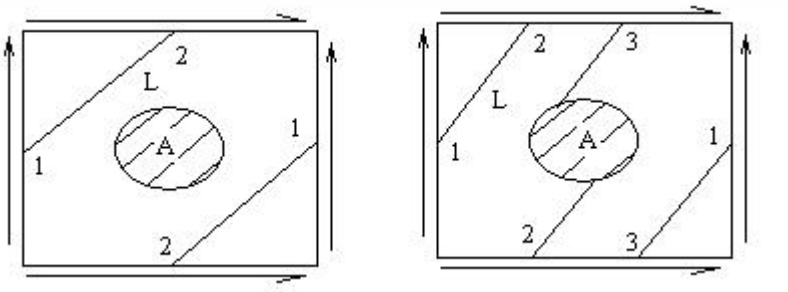



**Fig.**3.4

Similar to these results of Iseri, we obtained a result for Smarandache 2-manifolds in [9].

**Theorem** 3.1.([9]) There are Smarandache 2-manifolds in map geometries with or without

boundary, particularly,

(1) For a map $M$ on a 2-manifold with order$\geq 3$, vertex valency$\geq 3$ and a face $f \in F(M)$, there is an angle factor $\mu$ such that $(M, \mu)$ and $(M, \mu)^{-1}$ is a paradoxist geometry by denial the axiom (A5) with these axioms (A5), (L5) and (R5).

(2) There are non-geometries in map geometries with or without boundary.

(3) Unless axioms I-3, II-3, III-2, V-1 and V-2 in the Hilbert's axiom system for an Euclid geometry, an anti-geometry can be gotten from map geometries with or without boundary by denial other axioms in this axiom system.

(4) Unless the axiom (C3), a counter-projective geometry can be gotten from map geometries with or without boundary by denial axioms (C1) and (C2).

## §4. Constructing Smarandache n-manifolds

The constructions applied in map geometries can be generalized to differential $n$-manifolds for Smarandache $n$-manifolds, which also enables us to affirm that Smarandache geometries include nearly all existent differential geometries, such as Finsler geometry and Riemannian geometry, etc..

4.1. **Differentially Smarandache $n$-manifolds**

A differential $n$-manifold $(M^n, \mathcal{A})$ is an $n$-manifold $M^n, M^n = \bigcup\limits_{i\in I} U_i$, endowed with a $C^r$ differential structure $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)|\alpha \in I\}$ on $M^n$ for an integer $r$ with following conditions hold.

(1) $\{U_\alpha; \alpha \in I\}$ is an open covering of $M^n$;

(2) For $\forall \alpha, \beta \in I$, atlases $(U_\alpha, \varphi_\alpha)$ and $(U_\beta, \varphi_\beta)$ are *equivalent*, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the *overlap maps*

$$\varphi_\alpha\varphi_\beta^{-1} : \varphi_\beta(U_{\alpha\bigcap U_\beta}) \to \varphi_\beta(U_\beta) \text{ and } \varphi_\beta\varphi_\alpha^{-1} : \varphi_\beta(U_{\alpha\bigcap U_\beta}) \to \varphi_\alpha(U_\alpha)$$

are $C^r$;

(3) $\mathcal{A}$ is maximal, i.e., if $(U, \varphi)$ is an atlas of $M^n$ equivalent with one atlas in $\mathcal{A}$, then $(U, \varphi) \in \mathcal{A}$.

An $n$-manifold is smooth if it is endowed with a $C^\infty$ differential structure.

**Construction** 4.1 Let $M^n$ be an $n$-manifold with an atlas $\mathcal{A} = \{(U_p, \varphi_p)|p \in M^n\}$. For $\forall p \in M^n$ with a local coordinates $(x_1, x_2, \cdots, x_n)$, define a spatially directional mapping $\omega : p \to \mathbf{R}^n$ action on $\varphi_p$ by

$$\omega : p \to \varphi_p^\omega(p) = \omega(\varphi_p(p)) = (\omega_1, \omega_2, \cdots, \omega_n),$$

i.e., if a line $L$ passes through $\varphi(p)$ with direction angles $\theta_1, \theta_2, \cdots, \theta_n$ with axes $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ in $\mathbf{R}^n$, then its direction becomes

$$\theta_1 - \frac{\vartheta_1}{2} + \sigma_1, \theta_2 - \frac{\vartheta_2}{2} + \sigma_2, \cdots, \theta_n - \frac{\vartheta_n}{2} + \sigma_n,$$

after passing through $\varphi_p(p)$, where for any integer $1 \leq i \leq n$, $\omega_i \equiv \vartheta_i(mod 4\pi)$, $\vartheta_i \geq 0$ and

$$\sigma_i = \begin{cases} \pi, & if \ \ 0 \leq \omega_i < 2\pi, \\ 0, & if \ \ 2\pi < \omega_i < 4\pi. \end{cases}$$

A manifold $M^n$ endowed with such a spatially directional mapping $\omega : M^n \to \mathbf{R}^n$ is called an $n$-dimensional pseudo-manifold, denoted by $(M^n, \mathcal{A}^\omega)$.

**Definition** 4.1. A spatially directional mapping $\omega : M^n \to \mathbf{R}^n$ is euclidean if for any point $p \in M^n$ with a local coordinates $(x_1, x_2, \cdots, x_n)$, $\omega(p) = (2\pi k_1, 2\pi k_2, \cdots, 2\pi k_n)$ with $k_i \equiv 1(mod 2)$ for $1 \leq i \leq n$, otherwise, non-euclidean.

**Definition** 4.2. Let $\omega : M^n \to \mathbf{R}^n$ be a spatially directional mapping and $p \in (M^n, \mathcal{A}^\omega)$, $\omega(p)(\mod 4\pi) = (\omega_1, \omega_2, \cdots, \omega_n)$. Call a point $p$ elliptic, euclidean or hyperbolic in direction $\mathbf{e}_i$, $1 \leq i \leq n$ if $0 \leq \omega_i < 2\pi$, $\omega_i = 2\pi$ or $2\pi < \omega_i < 4\pi$.

Then we got serval results for Smarandache $n$-manifolds following.

**Theorem** 4.1.([14]) For a point $p \in M^n$ with local chart $(U_p, \varphi_p)$, $\varphi_p^\omega = \varphi_p$ if and only if $\omega(p) = (2\pi k_1, 2\pi k_2, \cdots, 2\pi k_n)$ with $k_i \equiv 1(\mod 2)$ for $1 \leq i \leq n$.

**Corollary** 4.1. Let $(M^n, \mathcal{A}^\omega)$ be a pseudo-manifold. Then $\varphi_p^\omega = \varphi_p$ if and only if every point in $M^n$ is euclidean.

**Theorem** 4.2.([14]) Let $(M^n, \mathcal{A}^\omega)$ be an $n$-dimensional pseudo-manifold and $p \in M^n$. If there are euclidean and non-euclidean points simultaneously or two elliptic or hyperbolic points in a same direction in $(U_p, \varphi_p)$, then $(M^n, \mathcal{A}^\omega)$ is a Smarandache $n$-manifold.

4.2. **Tangent and cotangent vector spaces**

The tangent vector space at a point of a smoothly Smarandache $n$-manifold is introduced in the following.

**Definition** 4.3. Let $(M^n, \mathcal{A}^\omega)$ be a smoothly differential Smarandache $n$-manifold and $p \in M^n$. A tangent vector $v$ at $p$ is a mapping $v : X_p \to \mathbf{R}$ with these following conditions hold.

(1) $\forall g, h \in X_p, \forall \lambda \in \mathbf{R}, \ v(h + \lambda h) = v(g) + \lambda v(h)$;

(2) $\forall g, h \in X_p, v(gh) = v(g)h(p) + g(p)v(h)$.

Denote all tangent vectors at a point $p \in (M^n, \mathcal{A}^\omega)$ by $T_pM^n$ and define addition "$+$" and scalar multiplication "$\cdot$" for $\forall u, v \in T_pM^n, \lambda \in \mathbf{R}$ and $f \in X_p$ by

$$(u + v)(f) = u(f) + v(f), \quad (\lambda u)(f) = \lambda \cdot u(f).$$

Then it can be shown immediately that $T_pM^n$ is a vector space under these two operations "$+$" and "$\cdot$" with basis determined in the next theorem.

**Theorem** 4.3.([14]) For any point $p \in (M^n, \mathcal{A}^\omega)$ with a local chart $(U_p, \varphi_p)$, $\varphi_p(p) = (x_1^i x_2^0, \cdots, x_n^0)$, if there are just $s$ euclidean directions along $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_s}$ for a point , then the dimension of $T_pM^n$ is

$$\dim T_pM^n \ = \ 2n - s$$

with a basis

$$\{\frac{\partial}{\partial x^{i_j}}|_p \mid 1 \leq j \leq s\} \bigcup \{\frac{\partial^-}{\partial x^l}|_p, \frac{\partial^+}{\partial x^l}|_p \mid 1 \leq l \leq n \text{ and } l \neq i_j, 1 \leq j \leq s\}.$$

The cotangent vector space at a point of $(M^n, \mathcal{A}^\omega)$ is defined in the next.

**Definition** 4.4. For $\forall p \in (M^n, \mathcal{A}^\omega)$, the dual space $T_p^*M^n$ is called a co-tangent vector space at $p$.

**Definition** 4.5. For $f \in \Im_p, d \in T_p^*M^n$ and $v \in T_pM^n$, the action of $d$ on $f$, called a differential operator $d : \Im_p \to \mathbf{R}$, is defined by

$$df \ = \ v(f).$$

Then we immediately got the basis of cotangent vector space at a point.

**Theorem** 4.4.([14]) For any point $p \in (M^n, \mathcal{A}^\omega)$ with a local chart $(U_p, \varphi_p)$, $\varphi_p(p) = (x_1^0, x_2^0, \cdots, x_n^0)$, if there are just $s$ euclidean directions along $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_s}$ for a point , then the dimension of $T_p^*M^n$ is

$$\dim T_p^*M^n \ = \ 2n - s$$

with a basis

$$\{dx^{i_j}|_p \mid 1 \leq j \leq s\} \bigcup \{d^-x_p^l, d^+x^l|_p \mid 1 \leq l \leq n \ and \ l \neq i_j, 1 \leq j \leq s\},$$

where

$$dx^i|_p(\frac{\partial}{\partial x^j}|_p) = \delta_j^i \ and \ d^{\epsilon_i}x^i|_p(\frac{\partial^{\epsilon_i}}{\partial x^j}|_p) = \delta_j^i,$$

for $\epsilon_i \in \{+, -\}, 1 \leq i \leq n$.

4.3. **Pseudo-manifold geometries**

Here we introduce Minkowski norms on these pseudo-manifolds $(M^n, \mathcal{A}^\omega)$.

**Definition** 4.6. A Minkowski norm on a vector space $V$ is a function $F : V \to \mathbf{R}$ such that

(1) $F$ is smooth on $V\backslash\{0\}$ and $F(v) \geq 0$ for $\forall v \in V$;

(2) $F$ is 1-homogenous, i.e., $F(\lambda v) = \lambda F(v)$ for $\forall \lambda > 0$;

(3) For all $y \in V\backslash\{0\}$, the symmetric bilinear form $g_y : V \times V \to \mathbf{R}$ with

$$g_y(u, v) = \sum_{i,j} \frac{\partial^2 F(y)}{\partial y^i \partial y^j}$$

is positive definite for $u, v \in V$.

Denote by $TM^n = \bigcup\limits_{p \in (M^n, \mathcal{A}^\omega)} T_pM^n$.

**Definition** 4.7. A pseudo-manifold geometry is a pseudo-manifold $(M^n, \mathcal{A}^\omega)$ endowed with a Minkowski norm $F$ on $TM^n$.

Then we found the following result.

**Theorem** 4.5.([14]) There are pseudo-manifold geometries.

4.4. **Principal fiber bundles and connections**

Although the dimension of each tangent vector space maybe different, we can also introduce principal fiber bundles and connections on pseudo-manifolds as follows.

**Definition** 4.8. A principal fiber bundle (PFB) consists of a pseudo-manifold $(P, \mathcal{A}_1^\omega)$, a projection $\pi : (P, \mathcal{A}_1^\omega) \to (M, \mathcal{A}_0^{\pi(\omega)})$, a base pseudo-manifold $(M, \mathcal{A}_0^{\pi(\omega)})$ and a Lie group $G$, denoted by $(P, M, \omega^\pi, G)$ such that (1), (2) and (3) following hold.

(1) There is a right freely action of $G$ on $(P, \mathcal{A}_1^\omega)$, i.e., for $\forall g \in G$, there is a diffeomorphism $R_g : (P, \mathcal{A}_1^\omega) \to (P, \mathcal{A}_1^\omega)$ with $R_g(p^\omega) = p^\omega g$ for $\forall p \in (P, \mathcal{A}_1^\omega)$ such that $p^\omega(g_1g_2) = (p^\omega g_1)g_2$ for

$\forall p \in (P, \mathcal{A}_1^{\omega})$, $\forall g_1, g_2 \in G$ and $p^{\omega}e = p^{\omega}$ for some $p \in (P^n, \mathcal{A}_1^{\omega})$, $e \in G$ if and only if $e$ is the identity element of $G$.

(2) The map $\pi : (P, \mathcal{A}_1^{\omega}) \to (M, \mathcal{A}_0^{\pi(\omega)})$ is onto with $\pi^{-1}(\pi(p)) = \{pg | g \in G\}$, $\pi\omega_1 = \omega_0\pi$, and regular on spatial directions of $p$, i.e., if the spatial directions of $p$ are $(\omega_1, \omega_2, \cdots, \omega_n)$, then $\omega_i$ and $\pi(\omega_i)$ are both elliptic, or euclidean, or hyperbolic and $|\pi^{-1}(\pi(\omega_i))|$ is a constant number independent of $p$ for any integer $i, 1 \leq i \leq n$.

(3) For $\forall x \in (M, \mathcal{A}_0^{\pi(\omega)})$ there is an open set $U$ with $x \in U$ and a diffeomorphism $T_u^{\pi(\omega)} : (\pi)^{-1}(U^{\pi(\omega)}) \to U^{\pi(\omega)} \times G$ of the form $T_u(p) = (\pi(p^{\omega}), s_u(p^{\omega}))$, where $s_u : \pi^{-1}(U^{\pi(\omega)}) \to G$ has the property $s_u(p^{\omega}g) = s_u(p^{\omega})g$ for $\forall g \in G, p \in \pi^{-1}(U)$.

**Definition** 4.9. Let $(P, M, \omega^{\pi}, G)$ be a PFB with $\dim G = r$. A subspace family $H = \{H_p | p \in (P, \mathcal{A}_1^{\omega}), \dim H_p = \dim T_{\pi(p)}M\}$ of $TP$ is called a connection if conditions (1) and (2) following hold.

(1) For $\forall p \in (P, \mathcal{A}_1^{\omega})$, there is a decomposition

$$T_pP = H_p \bigoplus V_p$$

and the restriction $\pi_*|_{H_p} : H_p \to T_{\pi(p)}M$ is a linear isomorphism.

(2) $H$ is invariant under the right action of $G$, i.e., for $p \in (P, \mathcal{A}_1^{\omega})$, $\forall g \in G$,

$$(R_g)_{*p}(H_p) = H_{pg}.$$

Then we obtained an interesting dimensional formula for $V_p$.

**Theorem** 4.6.([14]) Let $(P, M, \omega^{\pi}, G)$ be a $PFB$ with a connection $H$. $\forall p \in (P, \mathcal{A}_1^{\omega})$, if the number of euclidean directions of $p$ is $\lambda_P(p)$, then

$$\dim V_p = \frac{(\dim P - \dim M)(2\dim P - \lambda_P(p))}{\dim P}.$$

4.5. **Geometrical inclusions in Smarandache geometries**

We obtained geometrical theorems and inclusions in Smarandache geometries following.

**Theorem** 4.7.([14]) A pseudo-manifold geometry $(M^n, \varphi^{\omega})$ with a Minkowski norm on $TM^n$ is a Finsler geometry if and only if all points of $(M^n, \varphi^{\omega})$ are euclidean.

**Corollary** 4.2. There are inclusions among Smarandache geometries, Finsler geometry, Riemann geometry and Weyl geometry:

$$\{Smarandache\ geometries\} \supset \{pseudo - manifold\ geometries\}$$
$$\supset \{Finsler\ geometry\} \supset \{Riemann\ geometry\} \supset \{Weyl\ geometry\}.$$

**Theorem** 4.8.([14]) A pseudo-manifold geometry $(M_c^n, \varphi^{\omega})$ with a Minkowski norm on $TM^n$ is a Kähler geometry if and only if $F$ is a Hermite inner product on $M_c^n$ with all points of $(M^n, \varphi^{\omega})$ being euclidean.

**Corollary** 4.3. There are inclusions among Smarandache geometries, pseudo-manifold geometry and Kähler geometry:

$$\begin{aligned} \{Smarandache\ geometries\} &\supset \{pseudo - manifold\ geometries\} \\ &\supset \{K\ddot{a}hler\ geometry\}. \end{aligned}$$

# §5. Geometry on Combinatorial manifolds

The combinatorial speculation for geometry on manifolds enables us to consider these geometrical objects consisted by manifolds with different dimensions, i.e., combinatorial manifolds. Certainly, each combinatorial manifold is a Smarandache manifold itself. Similar to the construction of Riemannian geometry, by introducing metrics on combinatorial manifolds we can construct topological or differential structures on them and obtained an entirely new geometrical theory, which also convinces us those inclusions of geometries in Smarandache geometries established in Section 4 again.

For an integer $s \geq 1$, let $n_1, n_2, \cdots, n_s$ be an integer sequence with $0 < n_1 < n_2 < \cdots < n_s$. Choose $s$ open unit balls $B_1^{n_1}, B_2^{n_2}, \cdots, B_s^{n_s}$, where $\bigcap\limits_{i=1}^{s} B_i^{n_i} \neq \emptyset$ in $\mathbf{R}^{n_1+_2+\cdots n_s}$. Then a *unit open combinatorial ball of degree* $s$ is a union

$$\widetilde{B}(n_1, n_2, \cdots, n_s) = \bigcup_{i=1}^{s} B_i^{n_i}.$$

**Definition** 5.1. For a given integer sequence $n_1, n_2, \cdots, n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_s$, a combinatorial manifold $\widetilde{M}$ is a Hausdorff space such that for any point $p \in \widetilde{M}$, there is a local chart $(U_p, \varphi_p)$ of $p$, i.e., an open neighborhood $U_p$ of $p$ in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \to \widetilde{B}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$ with $\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\}$ and $\bigcup\limits_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\}$, denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ or $\widetilde{M}$ on the context and

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\},$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$. The maximum value of $s(p)$ and the dimension $\widehat{s}(p)$ of $\bigcap\limits_{i=1}^{s(p)} B_i^{n_i}$ are called the dimension and the intersectional dimensional of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ at the point $p$, respectively.

A combinatorial manifold $\widetilde{M}$ is called finite if it is just combined by finite manifolds.

A finite combinatorial manifold is given in Fig.5.1.

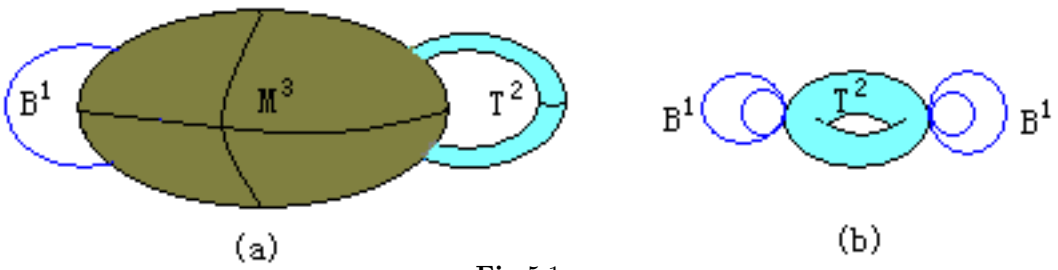



**Fig.**5.1

5.1. **Topological structures**

5.1.1. **Connectedness**

**Definition** 5.1. For two points $p, q$ in a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$, if there is a sequence $B_1, B_2, \cdots, B_s$ of $d$-dimensional open balls with two conditions following hold.

(1) $B_i \subset \widetilde{M}(n_1, n_2, \cdots, n_m)$ for any integer $i, 1 \leq i \leq s$ and $p \in B_1$, $q \in B_s$;

(2) The dimensional number $\dim(B_i \bigcap B_{i+1}) \geq d$ for $\forall i, 1 \leq i \leq s-1$.

Then points $p, q$ are called $d$-dimensional connected in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ and the sequence $B_1, B_2, \cdots, B_e$ a $d$-dimensional path connecting $p$ and $q$, denoted by $P^d(p, q)$.

If each pair $p, q$ of points in the finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is $d$-dimensional connected, then $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is called $d$-pathwise connected and say its connectivity$\geq d$.

Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a finitely combinatorial manifold and $d, d \geq 1$ an integer. We construct a labelled graph $G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ by

$$V(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]) = V_1 \bigcup V_2,$$

where

$V_1 = \{n_i - \text{manifolds } M^{n_i} \text{ in } \widetilde{M}(n_1, n_2, \cdots, n_m) | 1 \leq i \leq m\}$,

and

$V_2 = \{\text{isolated intersection points } O_{M^{n_i}, M^{n_j}} \text{ of} M^{n_i}, M^{n_j} \text{ in } \widetilde{M}(n_1, n_2, \cdots, n_m) \text{ for } 1 \leq i, j \leq m\}$.

Label $n_i$ for each $n_i$-manifold in $V_1$ and 0 for each vertex in $V_2$ and

$$E(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]) = E_1 \bigcup E_2,$$

where

$E_1 = \{(M^{n_i}, M^{n_j}) | \dim(M^{n_i} \bigcap M^{n_j}) \geq d, 1 \leq i, j \leq m\}$,

and

$E_2 = \{(O_{M^{n_i}, M^{n_j}}, M^{n_i}), (O_{M^{n_i}, M^{n_j}}, M^{n_j}) | M^{n_i} \text{ tangent } M^{n_j} \text{ at the point } O_{M^{n_i}, M^{n_j}} \text{ for } 1 \leq i, j \leq m\}$.

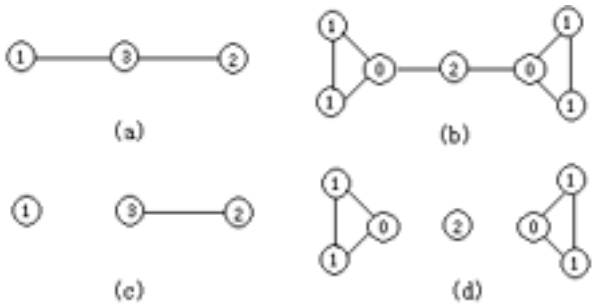



**Fig.**5.2

For example, these correspondent labelled graphs gotten from finitely combinatorial manifolds in Fig.5.1 are shown in Fig.5.2, in where $d = 1$ for (a) and (b), $d = 2$ for (c) and (d).

For a given integer sequence $1 \leq n_1 < n_2 < \cdots < n_m, m \geq 1$, denote by $\mathcal{H}^d(n_1, n_2, \cdots, n_m)$ all these finitely combinatorial manifolds $\widetilde{M}(n_1, n_2, \cdots, n_m)$ with connectivity$\geq d$, where $d \leq n_1$ and $\mathcal{G}(n_1, n_2, \cdots, n_m)$ all these connected graphs $G[n_1, n_2, \cdots, n_m]$ with vertex labels 0, $n_1$, $n_2$, $\cdots$, $n_m$ and conditions following hold.

(1) The induced subgraph by vertices labelled with 1 in $G$ is a union of complete graphs;

(2) All vertices labelled with 0 can only be adjacent to vertices labelled with 1.

Then we knew a relation between sets $\mathcal{H}^d(n_1, n_2, \cdots, n_m)$ and $\mathcal{G}(n_1, n_2, \cdots, n_m)$.

**Theorem** 5.1.([17]) Let $1 \leq n_1 < n_2 < \cdots < n_m, m \geq 1$ be a given integer sequence. Then every finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}^d(n_1, n_2, \cdots, n_m)$ defines a labelled connected graph $G[n_1, n_2, \cdots, n_m] \in \mathcal{G}(n_1, n_2, \cdots, n_m)$. Conversely, every labelled connected graph $G[n_1, n_2, \cdots, n_m] \in \mathcal{G}(n_1, n_2, \cdots, n_m)$ defines a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}^d(n_1, n_2, \cdots, n_m)$ for any integer $1 \leq d \leq n_1$.

5.1.2. **Homotopy**

Denoted by $f \simeq g$ two homotopic mappings $f$ and $g$. Following the same pattern of homotopic spaces, we define homotopically combinatorial manifolds in the next.

**Definition** 5.2. Two finitely combinatorial manifolds $\widetilde{M}(k_1, k_2, \cdots, k_l)$ and $\widetilde{M}(n_1, n_2, \cdots, n_m)$ are said to be homotopic if there exist continuous maps

$f : \widetilde{M}(k_1, k_2, \cdots, k_l) \to \widetilde{M}(n_1, n_2, \cdots, n_m)$,

$g : \widetilde{M}(n_1, n_2, \cdots, n_m) \to \widetilde{M}(k_1, k_2, \cdots, k_l)$,

such that $gf \simeq$ identity

$: \widetilde{M}(k_1, k_2, \cdots, k_l) \to \widetilde{M}(k_1, k_2, \cdots, k_l)$

and

$fg \simeq$ identity $: \widetilde{M}(n_1, n_2, \cdots, n_m) \to \widetilde{M}(n_1, n_2, \cdots, n_m)$.

Then we obtained the following result.

**Theorem** 5.2.([17]) Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $\widetilde{M}(k_1, k_2, \cdots, k_l)$ be finitely combinatorial manifolds with an equivalence $\varpi : G[\widetilde{M}(n_1, n_2, \cdots, n_m)] \to G[\widetilde{M}(k_1, k_2, \cdots, k_l)]$. If for $\forall M_1, M_2 \in V(G[\widetilde{M}(n_1, n_2, \cdots, n_m)])$, $M_i$ is homotopic to $\varpi(M_i)$ with homotopic mappings

$f_{M_i} : M_i \to \varpi(M_i)$, $g_{M_i} : \varpi(M_i) \to M_i$

such that

$f_{M_i}|_{M_i \bigcap M_j} = f_{M_j}|_{M_i \bigcap M_j}$, $\quad g_{M_i}|_{M_i \bigcap M_j} = g_{M_j}|_{M_i \bigcap M_j}$

providing $(M_i, M_j) \in E(G[\widetilde{M}(n_1, n_2, \cdots, n_m)])$ for $1 \leq i, j \leq m$, then $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is homotopic to $\widetilde{M}(k_1, k_2, \cdots, k_l)$.

5.1.3. **Fundamental $d$-groups**

**Definition** 5.3. Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a finitely combinatorial manifold. For an integer $d, 1 \leq d \leq n_1$ and $\forall x \in \widetilde{M}(n_1, n_2, \cdots, n_m)$, a fundamental $d$-group at the point $x$, denoted by $\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x)$ is defined to be a group generated by all homotopic classes of closed $d$-pathes based at $x$.

If $d = 1$ and $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is just a manifold $M$, we get that

$$\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x) = \pi(M, x).$$

Whence, fundamental $d$-groups are a generalization of fundamental groups in topology. We obtained the following characteristics for fundamental $d$-groups of finitely combinatorial manifolds.

**Theorem** 5.3.([17]) Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a $d$-connected finitely combinatorial manifold with $1 \leq d \leq n_1$. Then

(1) For $\forall x \in \widetilde{M}(n_1, n_2, \cdots, n_m)$,

$$\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x) \cong (\bigoplus_{M \in V(G^d)} \pi^d(M)) \bigoplus \pi(G^d),$$

where $G^d = G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$, $\pi^d(M), \pi(G^d)$ denote the fundamental $d$-groups of a manifold $M$ and the graph $G^d$, respectively and

(2) For $\forall x, y \in \widetilde{M}(n_1, n_2, \cdots, n_m)$,

$$\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x) \cong \pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), y).$$

A $d$-connected finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is said to be simply $d$-connected if $\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x)$ is trivial. As a consequence, we get the following result by Theorem 2.7.

**Corollary** 5.1. A $d$-connected finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is simply $d$-connected if and only if

(1) For $\forall M \in V(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)])$, $M$ is simply $d$-connected

and

(2) $G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ is a tree.

5.1.4. **Euler-Poincare characteristic**

The integer

$$\chi(\mathfrak{M}) = \sum_{i=0}^{\infty} (-1)^i \alpha_i,$$

with $\alpha_i$ the number of $i$-dimensional cells in a $CW$-complex $\mathfrak{M}$ is called the *Euler-Poincare characteristic* of the complex $\mathfrak{M}$. Now define a clique sequence $\{Cl(i)\}_{i\geq 1}$ in the graph $G[\widetilde{M}]$ by the following programming.

STEP 1. Let $Cl(G[\widetilde{M}]) = l_0$. Construct

$$\begin{aligned} Cl(l_0) &= \{K_1^{l_0}, K_2^{l_0}, \cdots, K_p^{i_0} | K_i^{l_0} \succ G[\widetilde{M}] \text{ and } K_i^{l_0} \cap K_j^{l_0} = \emptyset, \\ &\quad \text{or a vertex} \in \mathrm{V(G[\widetilde{M}])} \text{ for } i \neq j, 1 \leq i, j \leq p\}. \end{aligned}$$

STEP 2. Let $G_1 = \bigcup\limits_{K^l \in Cl(l)} K^l$ and $Cl(G[\widetilde{M}] \setminus G_1) = l_1$. Construct

$$\begin{aligned} Cl(l_1) &= \{K_1^{l_1}, K_2^{l_1}, \cdots, K_q^{i_1} | K_i^{l_1} \succ G[\widetilde{M}] \text{ and } K_i^{l_1} \cap K_j^{l_1} = \emptyset \\ &\quad \text{or a vertex} \in \mathrm{V(G[\widetilde{M}])} \text{ for } i \neq j, 1 \leq i, j \leq q\}. \end{aligned}$$

STEP 3. Assume we have constructed $Cl(l_{k-1})$ for an integer $k \geq 1$. Let

$$G_k = \bigcup_{K^{l_{k-1}} \in Cl(l)} K^{l_{k-1}}$$

and

$$Cl(G[\widetilde{M}] \setminus (G_1 \cup \cdots \cup G_k)) = l_k.$$

We construct

$$\begin{aligned} Cl(l_k) &= \{K_1^{l_k}, K_2^{l_k}, \cdots, K_r^{l_k} | K_i^{l_k} \succ G[\widetilde{M}] \text{ and } K_i^{l_k} \cap K_j^{l_k} = \emptyset, \\ &\quad \text{or a vertex} \in \mathrm{V(G[\widetilde{M}])} \text{ for } i \neq j, 1 \leq i, j \leq r\}. \end{aligned}$$

STEP 4. Continue STEP 3 until we find an integer $t$ such that there are no edges in $G[\widetilde{M}] \setminus \bigcup\limits_{i=1}^{t} G_i$.

By this clique sequence $\{Cl(i)\}_{i\geq 1}$, we calculated the Euler-Poincare characteristic of finitely combinatorial manifolds.

**Theorem** 5.4.([17]) Let $\widetilde{M}$ be a finitely combinatorial manifold. Then

$$\chi(\widetilde{M}) = \sum_{K^k\in Cl(k),k\geq 2}\ \sum_{M_{i_j}\in V(K^k),1\leq j\leq s\leq k} (-1)^{s+1}\chi(M_{i_1}\bigcap\cdots\bigcap M_{i_s}).$$

5.2. **Differential structures**

5.2.1. **Differentially combinatorial manifolds**

These differentially combinatorial manifolds are defined in next definition.

**Definition** 5.4. For a given integer sequence $1 \leq n_1 < n_2 < \cdots < n_m$, a combinatorially $C^h$ differential manifold $(\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$ is a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$, $\widetilde{M}(n_1, n_2, \cdots, n_m) = \bigcup\limits_{i\in I} U_i$, endowed with a atlas $\widetilde{\mathcal{A}} = \{(U_\alpha; \varphi_\alpha)|\alpha \in I\}$ on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ for an integer $h, h \geq 1$ with conditions following hold.

(1) $\{U_\alpha; \alpha \in I\}$ is an open covering of $\widetilde{M}(n_1, n_2, \cdots, n_m)$;

(2) For $\forall \alpha, \beta \in I$, local charts $(U_\alpha; \varphi_\alpha)$ and $(U_\beta; \varphi_\beta)$ are *equivalent*, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the *overlap maps*

$$\varphi_\alpha\varphi_\beta^{-1} : \varphi_\beta(U_\alpha\bigcap U_\beta) \to \varphi_\beta(U_\beta) \ \text{ and } \ \varphi_\beta\varphi_\alpha^{-1} : \varphi_\beta(U_\alpha\bigcap U_\beta) \to \varphi_\alpha(U_\alpha)$$

are $C^h$ mappings;

(3) $\widetilde{\mathcal{A}}$ is maximal, i.e., if $(U; \varphi)$ is a local chart of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ equivalent with one of local charts in $\widetilde{\mathcal{A}}$, then $(U; \varphi) \in \widetilde{\mathcal{A}}$.

Denote by $(\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$ a combinatorially differential manifold. A finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is said to be smooth if it is endowed with a $C^\infty$ differential structure.

5.2.2. **Tangent and cotangent vector spaces**

**Definition** 5.5. Let $(\widetilde{M}(n_1, n_2, \cdots, n_m), \widetilde{\mathcal{A}})$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. A tangent vector $v$ at $p$ is a mapping $v : X_p \to \mathbf{R}$ with conditions following hold.

(1) $\forall g, h \in X_p, \forall \lambda \in \mathbf{R},\ v(h + \lambda h) = v(g) + \lambda v(h)$;

(2) $\forall g, h \in X_p, v(gh) = v(g)h(p) + g(p)v(h)$.

Denoted all tangent vectors at $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$ by $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and define addition "$+$" and scalar multiplication "$\cdot$" for $\forall u, v \in T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$, $\lambda \in \mathbf{R}$ and $f \in X_p$ by

$$(u + v)(f) = u(f) + v(f), \quad (\lambda u)(f) = \lambda \cdot u(f).$$

Then it can be shown immediately that $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ is a vector space under these two operations "$+$" and "$\cdot$" with a basis determined in next theorem.

**Theorem** 5.5.([17]) For any point $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$ with a local chart $(U_p; [\varphi_p])$, the dimension of $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ is

$$\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum_{i=1}^{s(p)}(n_i - \widehat{s}(p)),$$

with a basis matrix

$$[\frac{\partial}{\partial \overline{x}}]_{s(p)\times n_{s(p)}} =$$

$$\left[\begin{array}{ccccccccc}
\frac{1}{s(p)}\frac{\partial}{\partial x^{11}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{1\widehat{s}(p)}} & \frac{\partial}{\partial x^{1(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{1n_1}} & \cdots & 0 \\
\frac{1}{s(p)}\frac{\partial}{\partial x^{21}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{2\widehat{s}(p)}} & \frac{\partial}{\partial x^{2(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{2n_2}} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
\frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)1}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)\widehat{s}(p)}} & \frac{\partial}{\partial x^{s(p)(\widehat{s}(p)+1)}} & \cdots & \cdots & \frac{\partial}{\partial x^{s(p)(n_{s(p)}-1)}} & \frac{\partial}{\partial x^{s(p)n_{s(p)}}}
\end{array}\right],$$

where $x^{il} = x^{jl}$ for $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$, namely there is a smoothly functional matrix $[v_{ij}]_{s(p)\times n_{s(p)}}$ such that for any tangent vector $\overline{v}$ at a point $p$ of $\widetilde{M}(n_1, n_2, \cdots, n_m)$,

$$\overline{v} = [v_{ij}]_{s(p)\times n_{s(p)}} \odot [\frac{\partial}{\partial \overline{x}}]_{s(p)\times n_{s(p)}},$$

where $[a_{ij}]_{k\times l} \odot [b_{ts}]_{k\times l} = \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{l} a_{ij}b_{ij}$.

**Definition** 5.6. For $\forall p \in (\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$, the dual space $T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ is called a co-tangent vector space at $p$.

**Definition** 5.7. For $f \in X_p, d \in T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $\overline{v} \in T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$, the action of $d$ on $f$, called a differential operator $d : X_p \to \mathbf{R}$, is defined by

$$df \ = \ \overline{v}(f).$$

Then we then obtained the result on the basis of cotangent vector space at a point following.

**Theorem** 5.6.([17]) For $\forall p \in (\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$ with a local chart $(U_p; [\varphi_p])$, the dimension of $T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ is

$$\dim T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum_{i=1}^{s(p)}(n_i - \widehat{s}(p)),$$

with a basis matrix

$$[d\overline{x}]_{s(p)\times n_{s(p)}} =$$

$$\left[\begin{array}{ccccccccc}
\frac{dx^{11}}{s(p)} & \cdots & \frac{dx^{1\widehat{s}(p)}}{s(p)} & dx^{1(\widehat{s}(p)+1)} & \cdots & dx^{1n_1} & \cdots & 0 \\
\frac{dx^{21}}{s(p)} & \cdots & \frac{dx^{2\widehat{s}(p)}}{s(p)} & dx^{2(\widehat{s}(p)+1)} & \cdots & dx^{2n_2} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
\frac{dx^{s(p)1}}{s(p)} & \cdots & \frac{dx^{s(p)\widehat{s}(p)}}{s(p)} & dx^{s(p)(\widehat{s}(p)+1)} & \cdots & \cdots & dx^{s(p)n_{s(p)}-1} & dx^{s(p)n_{s(p)}}
\end{array}\right],$$

where $x^{il} = x^{jl}$ for $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$, namely for any co-tangent vector $d$ at a point $p$ of $\widetilde{M}(n_1, n_2, \cdots, n_m)$, there is a smoothly functional matrix $[u_{ij}]_{s(p)\times s(p)}$ such that,

$$d = [u_{ij}]_{s(p)\times n_{s(p)}} \odot [d\overline{x}]_{s(p)\times n_{s(p)}}.$$

5.2.3. **Tensor fields**

**Definition** 5.8. Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. A tensor of type $(r, s)$ at the point $p$ on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is an $(r+s)$-multilinear function $\tau$,

$$\tau : \underbrace{T_p^*\widetilde{M} \times \cdots \times T_p^*\widetilde{M}}_{r} \times \underbrace{T_p\widetilde{M} \times \cdots \times T_p\widetilde{M}}_{s} \to \mathbf{R},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$.

Then we found the next result.

**Theorem** 5.7([17]) Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. Then

$$T_s^r(p, \widetilde{M}) = \underbrace{T_p\widetilde{M} \otimes \cdots \otimes T_p\widetilde{M}}_{r} \otimes \underbrace{T_p^*\widetilde{M} \otimes \cdots \otimes T_p^*\widetilde{M}}_{s},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$, particularly,

$$\dim T_s^r(p, \widetilde{M}) = (\widehat{s}(p) + \sum_{i=1}^{s(p)}(n_i - \widehat{s}(p)))^{r+s}.$$

5.2.4. **Exterior differentiations**

For the exterior differentiations on combinatorial manifolds, we find results following.

**Theorem** 5.8.([17]) Let $\widetilde{M}$ be a smoothly combinatorial manifold. Then there is a unique exterior differentiation $\widetilde{d} : \Lambda(\widetilde{M}) \to \Lambda(\widetilde{M})$ such that for any integer $k \geq 1$, $\widetilde{d}(\Lambda^k) \subset \Lambda^{k+1}(\widetilde{M})$ with conditions following hold.

(1) $\widetilde{d}$ is linear, i.e., for $\forall \varphi, \psi \in \Lambda(\widetilde{M})$, $\lambda \in \mathbf{R}$,

$$\widetilde{d}(\varphi + \lambda\psi) = \widetilde{d}\varphi \wedge \psi + \lambda\widetilde{d}\psi,$$

and for $\varphi \in \Lambda^k(\widetilde{M}), \psi \in \Lambda(\widetilde{M})$,

$$\widetilde{d}(\varphi \wedge \psi) = \widetilde{d}\varphi + (-1)^k\varphi \wedge \widetilde{d}\psi.$$

(2) For $f \in \Lambda^0(\widetilde{M})$, $\widetilde{d}f$ is the differentiation of $f$.

(3) $\widetilde{d}^2 = \widetilde{d} \cdot \widetilde{d} = 0$.

(4) $\widetilde{d}$ is a local operator, i.e., if $U \subset V \subset \widetilde{M}$ are open sets and $\alpha \in \Lambda^k(V)$, then $\widetilde{d}(\alpha|_U) = (\widetilde{d}\alpha)|_U$.

**Theorem** 5.9.([17]) Let $\omega \in \Lambda^1(\widetilde{M})$. Then for $\forall X, Y \in X(\widetilde{M})$,

$$\widetilde{d}\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

5.2.5. **Connections on combinatorial manifolds**

**Definition** 5.9. Let $\widetilde{M}$ be a smoothly combinatorial manifold. A connection on tensors of $\widetilde{M}$ is a mapping $\widetilde{D} : X(\widetilde{M}) \times T_s^r\widetilde{M} \to T_s^r\widetilde{M}$ with $\widetilde{D}_X\tau = \widetilde{D}(X, \tau)$ such that for $\forall X, Y \in X(\widetilde{M})$, $\tau, \pi \in T_s^r(\widetilde{M})$, $\lambda \in \mathbf{R}$ and $f \in C^\infty(\widetilde{M})$,

(1) $\widetilde{D}_{X+fY}\tau = \widetilde{D}_X\tau + f\widetilde{D}_Y\tau$; and $\widetilde{D}_X(\tau+\lambda\pi) = \widetilde{D}_X\tau + \lambda\widetilde{D}_X\pi$;

(2) $\widetilde{D}_X(\tau\otimes\pi) = \widetilde{D}_X\tau\otimes\pi + \sigma\otimes\widetilde{D}_X\pi$;

(3) For any contraction $C$ on $T^r_s(\widetilde{M})$,

$$\widetilde{D}_X(C(\tau)) = C(\widetilde{D}_X\tau).$$

Then we got results following.

**Theorem** 5.10.([17]) Let $\widetilde{M}$ be a smoothly combinatorial manifold. Then there exists a connection $\widetilde{D}$ locally on $\widetilde{M}$ with a form

$$(\widetilde{D}_X\tau)|_U = X^{\sigma\varsigma}\tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s),(\mu\nu)}\frac{\partial}{\partial x^{\mu_1\nu_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{\mu_r\nu_r}}\otimes dx^{\kappa_1\lambda_1}\otimes\cdots\otimes dx^{\kappa_s\lambda_s},$$

for $\forall Y\in X(\widetilde{M})$ and $\tau\in T^r_s(\widetilde{M})$, where

$$\begin{aligned}\tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s),(\mu\nu)} &= \frac{\partial\tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s)}}{\partial x^{\mu\nu}}\\ &+ \sum_{a=1}^{r}\tau^{(\mu_1\nu_1)\cdots(\mu_{a-1}\nu_{a-1})(\sigma\varsigma)(\mu_{a+1}\nu_{a+1})\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s)}\Gamma^{\mu_a\nu_a}_{(\sigma\varsigma)(\mu\nu)}\\ &- \sum_{b=1}^{s}\tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)\cdots(\kappa_{b-1}\lambda_{b-1})(\mu\nu)(\sigma_{b+1}\varsigma_{b+1})\cdots(\kappa_s\lambda_s)}\Gamma^{\sigma\varsigma}_{(\sigma_b\varsigma_b)(\mu\nu)},\end{aligned}$$

and $\Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)}$ is a function determined by

$$\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}}\frac{\partial}{\partial x^{\sigma\varsigma}} = \Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)}\frac{\partial}{\partial x^{\sigma\varsigma}},$$

on $(U_p;[\varphi_p]) = (U_p;x^{\mu\nu})$ of a point $p\in\widetilde{M}$, also called the coefficient on a connection.

**Theorem** 5.11.([17]) Let $\widetilde{M}$ be a smoothly combinatorial manifold with a connection $\widetilde{D}$. Then for $\forall X,Y\in X(\widetilde{M})$,

$$\widetilde{T}(X,Y) = \widetilde{D}_XY - \widetilde{D}_YX - [X,Y]$$

is a tensor of type $(1,2)$ on $\widetilde{M}$.

If $T(\frac{\partial}{\partial x^{\mu\nu}},\frac{\partial}{\partial x^{\sigma\varsigma}})\equiv 0$, we call $T$ torsion-free. This enables us getting the next useful result.

**Theorem** 5.12.([17]) A connection $\widetilde{D}$ on tensors of a smoothly combinatorial manifold $\widetilde{M}$ is torsion-free if and only if $\Gamma^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)} = \Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)}$.

5.2.6. **Combinatorially Finsler geometry**

**Definition** 5.10. A combinatorially Finsler geometry is a smoothly combinatorial manifold $\widetilde{M}$ endowed with a Minkowski norm $\widetilde{F}$ on $T\widetilde{M}$, denoted by $(\widetilde{M};\widetilde{F})$.

Then we got the following result.

**Theorem** 5.13.([17]) There are combinatorially Finsler geometries.

**Theorem** 5.14.([17]) A combinatorially Finsler geometry $(\widetilde{M}(n_1,n_2,\cdots,n_m);\widetilde{F})$ is a Smarandache geometry if $m\geq 2$.

Because combinatorially Finsler geometries are subsets of Smarandache geometries, we obtained the next consequence.

**Corollary** 5.2. There are inclusions among Smarandache geometries, Finsler geometry, Riemannian geometry and Weyl geometry:

$$\{Smarandache\ geometries\} \supset \{combinatorially\ Finsler\ geometries\}$$
$$\supset \{Finsler\ geometry\}\ and\ \{combinatorially\ Riemannian\ geometries\}$$
$$\supset \{Riemannian\ geometry\} \supset \{Weyl\ geometry\}.$$

5.2.7. **Integration on combinatorial manifolds**

For a smoothly combinatorial manifold $\widetilde{M}(n_1, \cdots, n_m)$, there must be an atlas $C = \{(\widetilde{U}_\alpha, [\varphi_\alpha])|\alpha \in \widetilde{I}\}$ on $\widetilde{M}(n_1, \cdots, n_m)$ consisting of positively oriented charts such that for $\forall \alpha \in \widetilde{I}$, $\widehat{s}(p)+\sum\limits_{i=1}^{s(p)}(n_i-\widehat{s}(p))$ is an constant $\widetilde{n}_{\widetilde{U}_\alpha}$ for $\forall p \in \widetilde{U}_\alpha$. Denote such atlas on $\widetilde{M}(n_1, \cdots, n_m)$ by $C_{\widetilde{M}}$ and an integer family $\mathcal{H}_{\widetilde{M}}(n, m) = \{n_{\widetilde{U}_\alpha}|\alpha \in \widetilde{I}\}$.

**Definition** 5.11. Let $\widetilde{M}$ be a smoothly combinatorial manifold with orientation $O$ and $(\widetilde{U}; [\varphi])$ a positively oriented chart with a constant $\widetilde{n} \in \mathcal{H}_{\widetilde{M}}(n, m)$. Suppose $\omega \in \Lambda^{\widetilde{n}_{\widetilde{U}}}(\widetilde{M}), \widetilde{U} \subset \widetilde{M}$ has compact support $\widetilde{C} \subset \widetilde{U}$. Then define

$$\int_{\widetilde{C}} \omega = \int \varphi_*(\omega|_{\widetilde{U}}).$$

Now if $C_{\widetilde{M}}$ is an atlas of positively oriented charts with an integer set $H_{\widetilde{M}}$, let $\widetilde{P} = \{(\widetilde{U}_\alpha, \varphi_\alpha, g_\alpha)|\alpha \in \widetilde{I}\}$ be a partition of unity subordinate to $C_{\widetilde{M}}$. For $\forall \omega \in \Lambda^n(\widetilde{M})$, $\widetilde{n} \in \mathcal{H}_{\widetilde{M}}(n, m)$, an integral of $\omega$ on $\widetilde{P}$ is defined by

$$\int_{\widetilde{P}} \omega = \sum_{\alpha \in \widetilde{I}} \int g_\alpha \omega.$$

**Definition** 5.12. Let $\widetilde{M}$ be a smoothly combinatorial manifold. A subset $D$ of $\widetilde{M}$ is with boundary if its points can be classified into two classes following.

**Class** 1 **(interior point Int**$D$**)** For $\forall p \in \mathrm{Int}D$, there is a neighborhood $V_p$ of $p$ enable $V_p \subset D$.

**Case** 2 **(boundary** $\partial D$**)** For $\forall p \in \partial D$, there is integers $\mu, \nu$ for a local chart $(U_p; [\varphi_p])$ of $p$ such that $x^{\mu\nu}(p) = 0$ but

$$U_p \bigcap D = \{q|q \in U_p, x^{\kappa\lambda} \geq 0 \text{ for } \forall \{\kappa, \lambda\} \neq \{\mu, \nu\}\}.$$

We then generalized the famous *Stokes' theorem* on manifolds in next theorem.

**Theorem** 5.15.([18]) Let $\widetilde{M}$ be a smoothly combinatorial manifold with an integer set $\mathcal{H}_{\widetilde{M}}(n, m)$ and $\widetilde{D}$ a boundary subset of $\widetilde{M}$. For $\widetilde{n} \in \mathcal{H}_{\widetilde{M}}$ if $\omega \in \Lambda^{\widetilde{n}}(\widetilde{M})$ has compact support, then

$$\int_{\widetilde{D}} d\omega = \int_{\partial \widetilde{D}} \omega,$$

with the convention $\int_{\partial \widetilde{D}} \omega = 0$ while $\partial \widetilde{D} = \emptyset$.

Corollaries following are immediately obtained by Theorem 5.15.

**Corollary** 5.3. Let $\widetilde{M}$ be a homogenously combinatorial manifold with an integer set $\mathcal{H}_{\widetilde{M}}(n,m)$ and $\widetilde{D}$ a boundary subset of $\widetilde{M}$. For $\widetilde{n} \in H_{\widetilde{M}}(n,m)$ if $\omega \in \Lambda^{\widetilde{n}}(\widetilde{M})$ has a compact support, then

$$\int_{\widetilde{D}} d\omega = \int_{\partial\widetilde{D}} \omega,$$

particularly, if $\widetilde{M}$ is nothing but a manifold, the Stokes theorem holds.

**Corollary** 5.4. Let $\widetilde{M}$ be a smoothly combinatorial manifold with an integer set $\mathcal{H}_{\widetilde{M}}(n,m)$. For $\widetilde{n} \in \mathcal{H}_{\widetilde{M}}(n,m)$, if $\omega \in \Lambda^{\widetilde{n}}(\widetilde{M})$ has a compact support, then

$$\int_{\widetilde{M}} \omega = 0.$$

# §6. Applications to other fields

6.1. **Applications to algebra**

The mathematical combinatorics can be also used to generalize algebraic systems, groups, rings, vector spaces, ... etc. in algebra as follows ([10] – [12]).

**Definition** 6.1 For any integers $n, n \geq 1$ and $i, 1 \leq i \leq n$, let $A_i$ be a set with an operation set $O(A_i)$ such that $(A_i, O(A_i))$ is a complete algebraic system. Then the union

$$\bigcup_{i=1}^{n} (A_i, O(A_i))$$

is called an $n$ multi-algebra system.

**Definition** 6.2 Let $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ be a complete multi-algebra system with a binary operation set $O(\widetilde{G}) = \{\times_i, 1 \leq i \leq n\}$. If for any integer $i, 1 \leq i \leq n$, $(G_i; \times_i)$ is a group and for $\forall x, y, z \in \widetilde{G}$ and any two binary operations "$\times$" and "$\circ$", $\times \neq \circ$, there is one operation, for example the operation $\times$ satisfying the distribution law to the operation "$\circ$" provided their operation results exist , i.e.,

$$x \times (y \circ z) = (x \times y) \circ (x \times z),$$

$$(y \circ z) \times x = (y \times x) \circ (z \times x),$$

then $\widetilde{G}$ is called a multi-group.

**Definition** 6.3. Let $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ be a complete multi-algebra system with a double binary operation set $O(\widetilde{R}) = \{(+_i, \times_i), 1 \leq i \leq m\}$. If for any integers $i, j,\ i \neq j, 1 \leq i, j \leq m$, $(R_i; +_i, \times_i)$ is a ring and for $\forall x, y, z \in \widetilde{R}$,

$$(x +_i y) +_j z = x +_i (y +_j z), \quad (x \times_i y) \times_j z = x \times_i (y \times_j z),$$

and

$$x \times_i (y +_j z) = x \times_i y +_j x \times_i z, \quad (y +_j z) \times_i x = y \times_i x +_j z \times_i x,$$

provided all these operation results exist, then $\widetilde{R}$ is called a multi-ring. If for any integer $1 \leq i \leq m$, $(R; +_i, \times_i)$ is a filed, then $\widetilde{R}$ is called a multi-filed.

**Definition** 6.4. Let $\widetilde{V} = \bigcup_{i=1}^{k} V_i$ be a complete multi-algebra system with a binary operation set $O(\widetilde{V}) = \{(\dot{+}_i, \cdot_i) \mid 1 \leq i \leq m\}$ and $\widetilde{F} = \bigcup_{i=1}^{k} F_i$ a multi-filed with a double binary operation set $O(\widetilde{F}) = \{(+_i, \times_i) \mid 1 \leq i \leq k\}$. If for any integers $i, j,\ 1 \leq i, j \leq k$ and $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \widetilde{V}$, $k_1, k_2 \in \widetilde{F}$,

$(i)$ $(V_i; \dot{+}_i, \cdot_i)$ is a vector space on $F_i$ with vector additive $\dot{+}_i$ and scalar multiplication $\cdot_i$;

$(ii)$ $(\mathbf{a}\dot{+}_i\mathbf{b})\dot{+}_j\mathbf{c} = \mathbf{a}\dot{+}_i(\mathbf{b}\dot{+}_j\mathbf{c})$;

$(iii)$ $(k_1 +_i k_2) \cdot_j \mathbf{a} = k_1 +_i (k_2 \cdot_j \mathbf{a})$;

provided all those operation results exist, then $\widetilde{V}$ is called a multi-vector space on the multi-filed $\widetilde{F}$ with a binary operation set $O(\widetilde{V})$, denoted by $(\widetilde{V}; \widetilde{F})$.

Elementary structural results for these multi-groups, multi-rings, multi-vector spaces,... can be found in references [9] – [13].

6.2. **Applications to theoretical physics**

Some physicists had applied Smarandache multi-spaces to solve many world problem by conservation laws, such as works in [2]. In fact, although the Bag Bang model is an application of the Einstein's gravitational equation to the universe, it throughout persists in the uniqueness of universes since one can not see other things happening in the spatial beyond the visual sense of mankind. This situation have been modified by physicists in theoretical physics such as those of gauge theory and string/M-theory adhered to a microspace at each point ([3]).

According the geometrical theory established in the last section, we can also introduce curvature tensors $R_{(\alpha\beta)(\mu\nu)}$ on smoothly combinatorial manifolds in the following way.

**Definition** 6.1. Let $\widetilde{M}$ be a smoothly combinatorial manifold with a connection $\widetilde{D}$. For $\forall X, Y, Z \in X(\widetilde{M})$, define a combinatorially curvature operator $\widetilde{\mathcal{R}}(X, Y) : X(\widetilde{M}) \to X(\widetilde{M})$ by

$$\widetilde{\mathcal{R}}(X, Y)Z = \widetilde{D}_X\widetilde{D}_Y Z - \widetilde{D}_Y\widetilde{D}_X Z - \widetilde{D}_{[X,Y]}Z,$$

and a combinatorially curvature tensor

$\widetilde{\mathcal{R}} : X(\widetilde{M}) \times X(\widetilde{M}) \times X(\widetilde{M}) \to X(\widetilde{M})$ by $\widetilde{\mathcal{R}}(Z, X, Y) = \widetilde{\mathcal{R}}(X, Y)Z$.

Then at each point $p \in \widetilde{M}$, there is a type $(1, 3)$ tensor $\widetilde{\mathcal{R}}_p : T_p\widetilde{M} \times T_p\widetilde{M} \times T_p\widetilde{M} \to T_p\widetilde{M}$ determined by $\widetilde{\mathcal{R}}(w, u, v) = \widetilde{\mathcal{R}}(u, v)w$ for $\forall u, v, w \in T_p\widetilde{M}$. Now let $(U_p; [\varphi_p])$ be a local chart at the point $p$, applying Theorems 5.5 and 5.6, we can find that

$$\widetilde{\mathcal{R}}(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}})\frac{\partial}{\partial x^{\sigma\varsigma}} = \widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}\frac{\partial}{\partial x^{\eta\theta}},$$

where

$$\widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)} = \frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\mu\nu}} - \frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\kappa\lambda}} + \Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\eta\theta}_{(\vartheta\iota)(\mu\nu)} - \Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\eta\theta}_{(\vartheta\iota)(\kappa\lambda)},$$

and $\Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)} \in C^\infty(U_p)$ determined by

$$\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}} \frac{\partial}{\partial x^{\kappa\lambda}} = \Gamma^{\sigma\varsigma}_{(\kappa\lambda)(\mu\nu)} \frac{\partial}{\partial x^{\sigma\varsigma}}.$$

Now we define $\widetilde{\mathcal{R}}_{(\mu\nu)(\kappa\lambda)} = \widetilde{\mathcal{R}}_{(\kappa\lambda)(\nu\mu)} = \widetilde{\mathcal{R}}^{\sigma\varsigma}_{(\mu\nu)(\sigma\varsigma)(\kappa\lambda)}$ and $R = g^{(\kappa\lambda)(\mu\nu)}\widetilde{\mathcal{R}}_{(\kappa\lambda)(\nu\mu)}$.

Then similar to the establishing of Einstein's gravitational equation, we know that

$$\widetilde{\mathcal{R}}_{(\mu\nu)(\kappa\lambda)} - \frac{1}{2}Rg_{(\mu\nu)(\kappa\lambda)} = -8\pi G T_{(\mu\nu)(\kappa\lambda)},$$

if we take smoothly combinatorial manifolds to describe the spacetime. Thereby there are Smarandache multi-space solutions in the Einstein's gravitational equation, particularly, solutions of combinatorially Euclidean spaces. For example, let

$$d\Omega^2(r,\theta,\varphi) = \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Then we can choose a multi-time system $\{t_1, t_2, \cdots, t_n\}$ to get a cosmic model of $n, n \geq 2$ combinatorially $\mathbf{R}^4$ spaces with line elements

$$ds_1^2 = -c^2dt_1^2 + a^2(t_1)d\Omega^2(r,\theta,\varphi),$$

$$ds_2^2 = -c^2dt_2^2 + a^2(t_2)d\Omega^2(r,\theta,\varphi),$$

$$\cdots\cdots,$$

$$ds_n^2 = -c^2dt_n^2 + a^2(t_n)d\Omega^2(r,\theta,\varphi).$$

As a by-product for the universe $\mathbf{R}^3$, there are maybe $n-1$ beings in the universe with different time system two by two for an integer $n \geq 2$ not alike that of humanity. So it is very encouraging for scientists looking for those beings in theory or experiments.

# References

[1] R. Abraham and Marsden, Foundation of Mechanics(2nd edition), Addison-Wesley, Reading, Mass, 1978.

[2] G. Bassini and S.Capozziello, Multi-spaces and many worlds from conservation laws, Progress in Physics, **4**(2006), 65-72.

[3] M. Carmeli, Classical Fields–General Relativity and Gauge Theory, World Scientific, 2001.

[4] W. H. Chern and X. X. Li, Introduction to Riemannian Geometry, Peking University Press, 2002.

[5] H. Iseri, Smarandache manifolds, American Research Press, Rehoboth, NM, 2002.

[6] L. Kuciuk and M. Antholy, An Introduction to Smarandache Geometries, JP Journal of Geometry and Topology, **5**(2005), No.1, 77-81.

[7] L. F. Mao, Automorphism Groups of Maps, Surfaces and Smarandache Geometries, American Research Press, 2005.

[8] L. F. Mao, On Automorphisms groups of Maps, Surfaces and Smarandache geometries, Sientia Magna, **1**(2005), No. 2, 55-73.

[9] L. F. Mao, Smarandache multi-space theory, Hexis, Phoenix, AZ，2006.

[10] L. F. Mao, On multi-metric spaces, Scientia Magna, **2**(2006), No.1, 87-94.

[11] L. F. Mao, On algebraic multi-group spaces, Scientia Magna, **2**(2006), No.1, 64-70.

[12] L. F. Mao, On algebraic multi-ring spaces, Scientia Magna, **2**(2006), No. 2, 48-54.

[13] L. F. Mao, On algebraic multi-vector spaces, Scientia Magna, **2**(2006), No. 2, 1-6.

[14] L. F. Mao, Pseudo-Manifold Geometries with Applications, e-print, arXiv: math. GM/0610307.

[15] L. F. Mao, A new view of combinatorial maps by Smarandache's notion, Selected Papers on Mathematical Combinatorics(I), 49-74, World Academic Union, 2006.

[16] L. F. Mao, Combinatorial speculation and the combinatorial conjecture for mathematics, Selected Papers on Mathematical Combinatorics(I), 1-22, World Academic Union, 2006.

[17] L. F. Mao, Geometrical theory on combinatorial manifolds, *JP J. Geometry and Topology*, **7**(2007), No.1, 65-114.

[18] L. F. Mao, A generalization of Stokes theorem on combinatorial manifolds, e-print, arXiv: math. GM/0703400.

[19] W. S. Massey, Algebraic topology, an introduction, Springer-Verlag, New York, etc., 1977.

[20] V. V. Nikulin and I.R.Shafarevlch, Geometries and Groups, Springer-Verlag Berlin Heidelberg, 1987.

[21] Joseph J. Rotman, An introduction to algebraic topology, Springer-Verlag New York Inc., 1988.

[22] F. Smarandache, Mixed noneuclidean geometries, eprint arXiv: math.GM/0010119, 10/2000.

[23] F. Smarandache, A unifying field in logic: neutrosophic field, Multi-value Logic, **8**(2002), No.3, 385-438.

[24] J. Stillwell, Classical topology and combinatorial group theory, Springer-Verlag New York Inc., 1980.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 81-92**

# Smarandache stepped functions

Mircea Eugen Selariu

National Research Institute for Electrochemistry and Condensed Matter
The "Politehnica" University of Timioara, Romania

**Abstract** The discovery of mathematical complements, assembled under the name of the eccentric mathematics, gave the opportunity for a series of applications, amongst which, in this article, are presented the impulse, step, and unitary ramp functions. The difference, in comparison with the same classic functions, from the distributions theory, is that those eccentric are periodical with a $2\pi$ period. By combining these between them, new mathematical functions have been defined; united under the name **Smarandache stepped functions**.

## §1. Introduction

Romanian mathematician Octavian Stănăsilă? sustains that the physics became a science when the calculus (mathematical analysis) has been discovered. In turn, the physics' development imposed the calculus' development.

The theoretical physics, and especially, the quantum mechanics, optics, wave propagation, different electromagnetism phenomena, and the solving of certain limit problems, imposed the introduction of new notions, which are not confined anymore to classical calculus (mathematical analysis), and whose justification could not be made within this frame [6]. This does not mean that it will not come a moment, in mathematics, when this thing can be done. It consists in the discovery of some mathematical complements, included in eccentric mathematics **EM**[8], [9], [10], [11], [12] etc., which extend at infinitum all current mathematical forms and objects, ensuring a vast extension of classical/ordinary mathematics, which will be named centric mathematics **CM**. The reunion of this two mathematics forms what is called the supermathematics **SM**.

## §2. The representation of derivatives of some functions

The fact that not every continue function is derivable, having as consequence the inexistence of velocity of a material point, in every moment of its movement, which, evidently, does not correspond to the reality, constitutes a sever difficulty in the **CM**which affects the unity and the generalization of the results, which is not the case in the **SM**.

For example, let's consider the first nowhere-derivable function, presented by Weierstrass [7, p. 105]:

(1) $W(t) = \sum\limits_{n=0}^{\infty} a^n \cos(b^n \pi t)$, $0 < a < 1$ and $b = 1, 3, 5, \cdots, (2n-1)$, an odd integer, such that

$$a,\ b > 1 + \frac{3\pi}{2} = 5,712.$$

A modification of Weierstrass' example will be obtained by the substitution in equation (1) of $\cos \pi\ t$ with linear Euler spline $E(t)$, which interpolates the argument $\cos \pi\ t$ in all integer values of $t$, and we obtain the graph [7] from figure 1. Next to it, it was presented the eccentric supermathematics functions family, of eccentric variable $\theta \equiv t$, named *bex t*, and which is a component/term of the eccentric amplitude function ($aex\theta$), defined by the relation

$$(2)\quad \begin{array}{l} \alpha(\theta) = aex\theta = \theta, \\ \beta(\theta) = \theta, \\ bex\theta = 0, \\ \arcsin[s \cdot \sin(\theta - \varepsilon)], \end{array}$$

where $\theta$ is the eccentric variable or the angle that a positive semi straight line, revolving around the excenter $S(s,\ \varepsilon)$ - or solar point, (Kepler affirmed that planets rotate around the Sun on circular orbits, but the Sun is not in the center of the orbits) - it makes it with $Ox$ axis [8], [9], [10], and $\alpha$ is the centric variable or the circular arc, of the unity circle, from the origin of the arc $A(1, 0)$ to a current point on circle $W(1, \alpha) \equiv W(r = rex\theta, \theta)$ the unitary eccentricity is $s = \frac{e}{R}$, or the distance between $S$ and $O$, and excenter $S$ or $E$ are ejected from the center $O$ on the $\varepsilon$direction.

For $\theta \Rightarrow \pi\ t$ and a phase difference $\varepsilon = -\frac{\pi}{2}$ will obtain the function or, more precisely, the functions family.

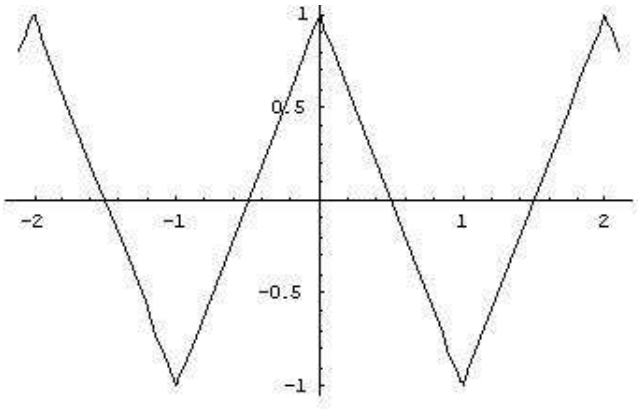



Fig. 1. Modified Weierstrass' function

(3) $bex\ t = \arcsin\left[s \sin\left(\pi \cdot t + \frac{\pi}{2}\right)\right]$,

whose graphs, of the numeric eccentricity $s \in [0, 1]$, with the step 0.1, are presented in figure 2.

It can be observed, without difficulty, that for $s = 0 \rightarrow aex\ \mathrm{t} = 0$ and for$s = 1$, the maximum limit (in graphs) of $s$, we obtain the graph of a function in "symmetric triangle teeth" (Fig. 1).

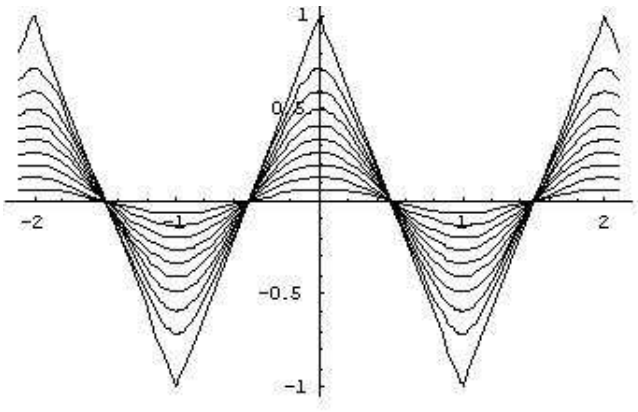



Fig. 2. The Eccentric SM function bex t

Because, the derivative of the function $aex\ t$ is the eccentric derivative function $dex\theta$:

(4) $\frac{d(aex\ t)}{dt} = \frac{d\alpha}{d\theta} = dex\ \theta = 1 - \frac{s\cdot\cos(\theta-\varepsilon)}{\sqrt{1-s^2\sin^2(\theta-\varepsilon)}}$,

it results that the second term from the relation (4) is exactly the derivative of the function $bex\ \theta$, that is:

(5) $\frac{d(bex\ \theta)}{d\theta} = \frac{s\cdot\cos(\theta-\varepsilon)}{\sqrt{1-s^2\sin^2(\theta-\varepsilon)}} = s\cdot coq\left(\pi\cdot t+\frac{\pi}{2}\right) = -s\left(\sin\pi\cdot t\right)$,

which is the product between the numerical eccentricity $s$ and the quadrilob cosine function $coq\theta$ [12], with a phase difference $\varepsilon = -\frac{\pi}{2}$, therefore it results $-s\cdot siq\theta$, whose graphs family are presented in the figure 4, for $s\in[0,1]$, with the step 0.1 and in the figure 3, for $s=1$.

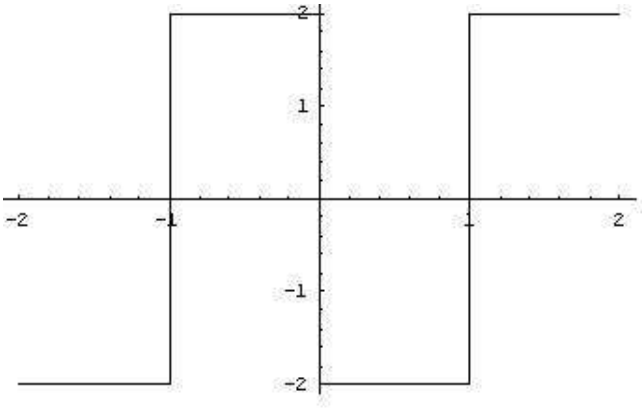



Fig. 3. Modified derivative of function Weierstrass

The quadrilob sinus function ($siq\theta$), for the numerical eccentricity $s=1$, represents, in the signals theory, the response of a relay to a sinusoidal signal; this function is also called **square sinus** [13, p. 31], which is exactly the eccentric sinus trigonometric function, with the numerical eccentricity $s=1$, defined on a square, non-rotated with $\frac{\pi}{4}$, as in the case of Alaci quadratic functions [12].

Corroborating the functions and their derivatives, it can be observed that they correspond between them. Thus, modified Weierstrass function, from figure 1, viewed as a $bex\ t$ function of numerical eccentricity $s=1$, becomes complete derivable.

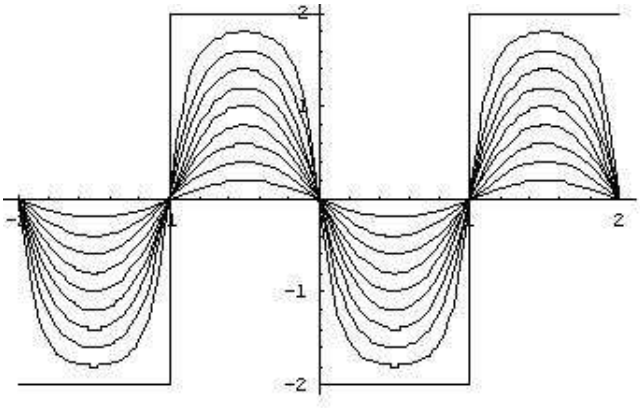



Fig. 4. The derivatives of the function bex 0

## §3. About distributions

In 1926 P. A. M. Dirac introduced, in the quantum mechanics, the delta "function" ($\delta$), which is over all null, with the exception of a point (in origin taking $\infty$ value), defined as follows:

$$(6)\ \delta(x) \stackrel{d}{=}: \begin{cases} 0,\ t \neq 0 \quad , \\ +\infty,\ t = 0, \end{cases}$$

and whose integral is:

$$(7)\ \int_{-\infty}^{\infty} \delta(x)dx = 1.$$

The same value of the integral is also for the unitary impulse function $\Delta(x, \lambda)$ defined by

$$(8)\ \Delta(x,\lambda) = \begin{cases} 0, & x < -\frac{\lambda}{2}, \\ \frac{1}{\lambda}, & -\frac{\lambda}{2} \leq x \leq \frac{\lambda}{2}, \\ 0, & x > \frac{\lambda}{2}. \end{cases}$$

It can be observed that for $\lambda \to 0$ we obtain the Dirac function $\delta(x)$. It must be mentioned that a rigorous definition of Dirac's impulse can be given within distributions theory [6] or of generalized functions, a chapter of functional analysis.

The unitary impulse can be viewed also as the derivative of the (ideal) unitary step function, or as of Heaviside function $\Gamma(x)$, defined as:

$$(9)\ \Gamma(x) =: \begin{cases} 0,\ x < 0, \\ 1,\ x > 0, \end{cases}$$

admitting, in this way, the derivability of any continue function on sections.

The unitary ramp function is defined as:

$$(10)\ R(x) =: \begin{cases} 0, x\ < 0, \\ x,\ x \geq 0, \end{cases}$$

and its derivative is the ideal unitary step function (Heaviside).

## §4. Periodical unitary step, impulse and ramp functions expressed as eccentric circular supermathematics functions (EC-SMF) and with eccentric quadrilob supermathematics functions (QL-SMF)

In figures 5 and 6 are presented the graphs of the eccentric cosine functions (*cex t*) and eccentric quadrilob cosine (*coq t*) respectively [12] for super-unitary numerical eccentricities s.

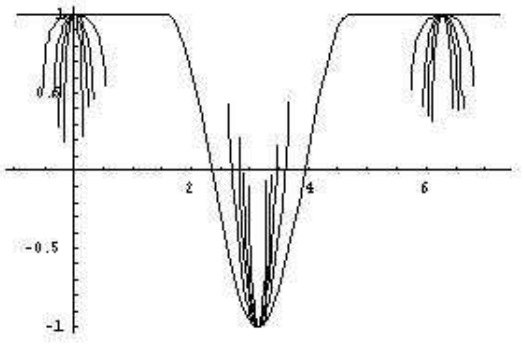



Fig. 5a. The function cex t,

for s = 1,2,3,4 and 6

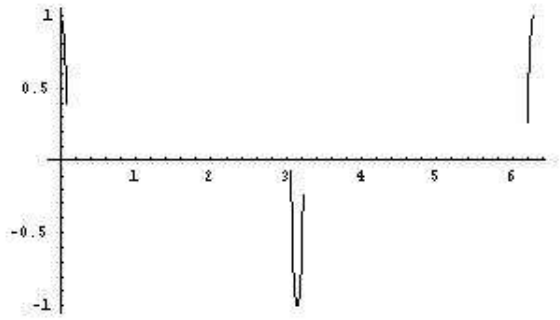



Fig. 5b. The function cex t,

for s = 4 π = 12,566

It can be observed, in the same time with the increase of the numerical eccentricity value $s$, the functions existence domain becomes restricted to the interval where a line, revolving from the excenter $S(s,\ \varepsilon)$, external to unity circle, intersects the unity circle. This interval I is periodical, with the period of $2\pi$ and it is defined by relation (9).

(11) $I = t_{final} - t_{initial} = 2\gamma = 2\arcsin\left(\frac{1}{s}\right)$,

for which the function

(12) $del\theta = \sqrt{1 - s^2\sin^2(\theta - \varepsilon)} = 0$,

where,

(13) $t_{initial} = \pi + \varepsilon - \gamma = \pi$ and $t_{final} = \pi + \varepsilon + \gamma = \pi$,

the eccentrical variable: $t \equiv \theta(\mathrm{mod}2\pi)$ such that, for the excenter $S$ going, on the $x$ axis ($\varepsilon = 0$), to infinite ($s \to \infty$), the domain$I$ goes to zero ($I \to 0$). From (12) it results that at $t = \pi$ and $s \to \infty$, the function $cex_1 t$ is an impulse signal of amplitude –1, that periodically repeats with a $2\pi$ period, and the second determination - with index 2 – of the function, $cex_2 t = 1$ for $t = 0 + 2k\pi$, $(k = 0, 1, 2, ...)$, therefore also at $t = 2\pi$, for $s \to \infty$, as it results also from the figures 5, A and 5, B.

We will call, these functions "**periodical impulse functions cext of unity amplitude** with $s \to \infty$". For $\varepsilon = \frac{\pi}{2}$, analogously, for $s \to \infty$, we obtain "**periodical impulse functions sext of unity amplitude".**

Because

(14) $cex_{1,2}^2\theta + sex_{1,2}^2\theta = 1$, where $cex_{1,2}t = \pm 1 \rightarrow sex_{1,2}t = 0$ and vice versa.

Therefore, at $t = \pi$, the function $sex_1 t = 0$ with the period $2\pi$ and the function

$I(t,s) = \frac{1}{sex_1 t} \Rightarrow \infty$, obtaining periodical unitary impulse functions, of an infinite amplitude, similar to Dirac's function, the difference being that it is periodic with a $2\pi$ period.

Also the quadrilob cosine function [12]

(15) $coq_1 t = \frac{\cos\theta}{\sqrt{1-s^2\sin^2(\theta-\varepsilon)}}$ $(\theta \equiv t)$, for $s \rightarrow \infty$ has at $t = \pi \pm 2k\pi$, $(k = 0, 1, 2, ...)$ the denominator *del* $t = 0$ and $\cos 0 = 1$, such that the amplitude goes to infinite and, this way, will obtain, again, a periodical unitary impulse function (Figure 6).

A periodical rectangular function of unity amplitude (Figure 6a) is given by the supermathematics eccentric quadrilob function:

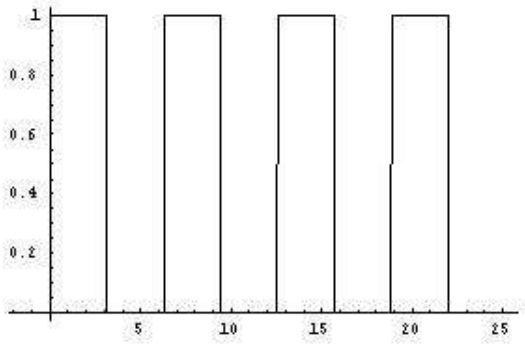


Fig. 6a

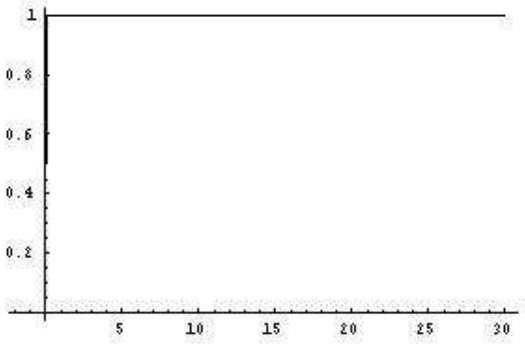


Fig. 6b

$0,5(1 - siq\ t)$, with a phase difference $\pi$

(16) $D(t,s) = \begin{cases} \frac{1}{2}\left[1 - \frac{\sin(t+\pi)}{\sqrt{1-s^2\cos^2(t+\pi)}}\right] & , \ t \geq 0, \\ 0 & , \ t < 0, \end{cases}$

which can be named periodical unitary step function, if the numerical eccentricity $s = 1$.

If $t \rightarrow \frac{t}{10}$, (Figure 6b), the first step extends from $\pi$ to $10\pi$. It results that for $t \rightarrow \frac{t}{\infty} \rightarrow 0$ it will be obtained a unitary step function on all axis $t > 0$.

An analogous function can be obtained also with the eccentric derivative function dex t of $s = 1$ (Figure 7a) and with $t \rightarrow \frac{t}{10}$ and $\varepsilon = -\pi$ (Figure 7b).

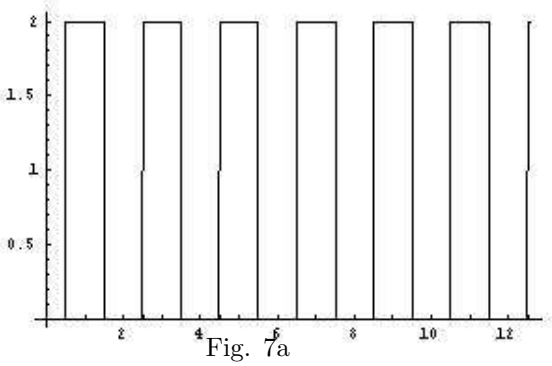


Fig. 7a

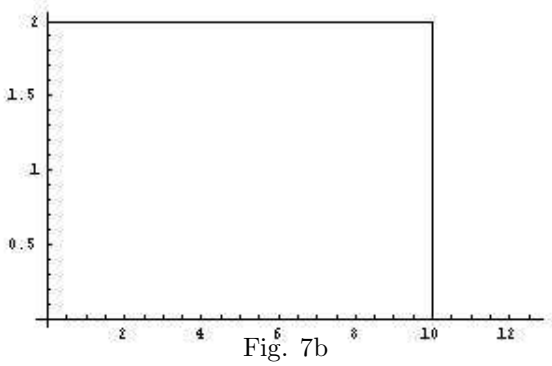


Fig. 7b

An ideal unitary ramp function can be obtained as a straight line passing through origin, of an angular coefficient m equal with unity ($m = 1$).

$$(17)\ y = \begin{cases} mx, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

A real unitary ramp function, that will admit certain aberrations from linearity, can also be obtained as a twisted [13] which passes through the origin $O(0, 0)$.

A twisted family, obtained for $s \neq 0$ in the interval $s \in [-1, 1]$, are presented in figure 8a, where, for $s = 0$, will obtain a ramp for $t \in [0,\ \infty]$.

Unitary ramp functions can be obtained by the substitution of the constant $m = \tan\alpha$ with the variable $m = tex\theta$ for a unitary eccentricity $s = 0, 1$.

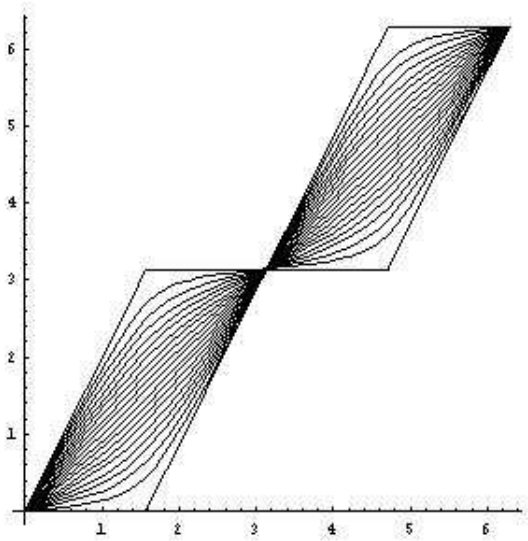


Fig. 8a

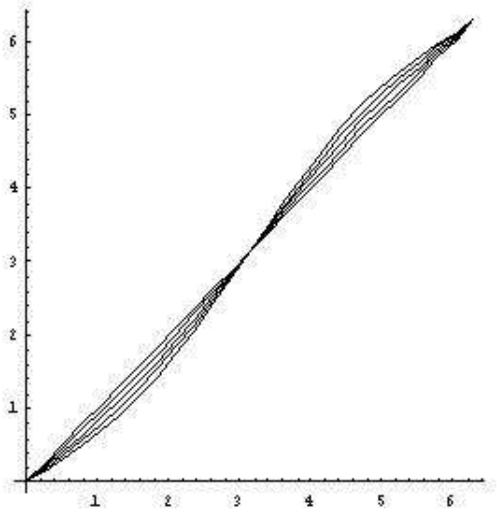
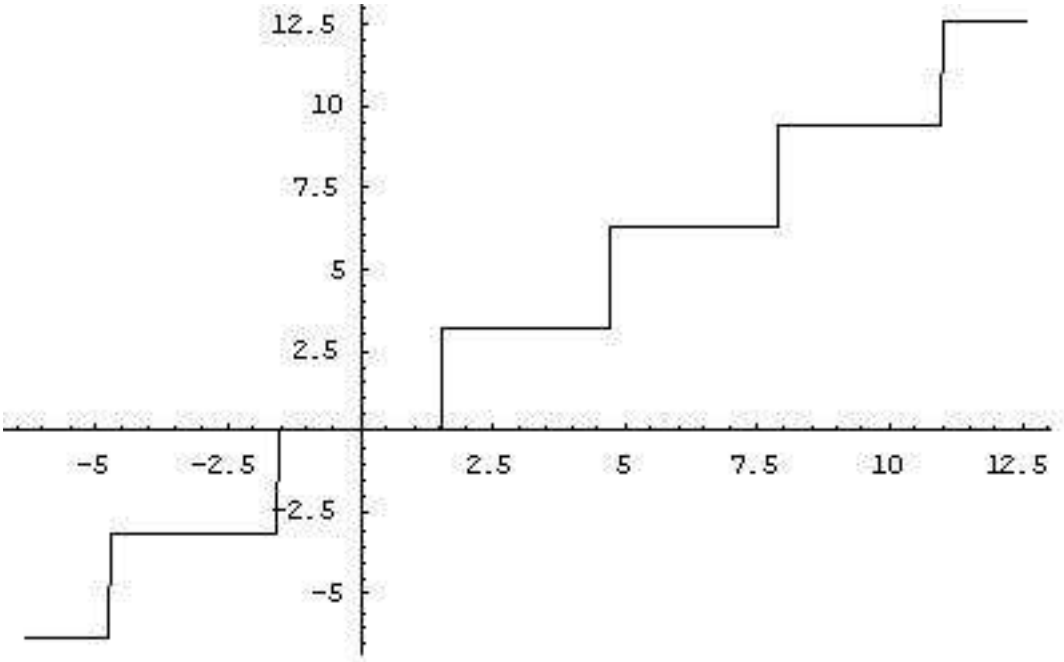


Fig.8b

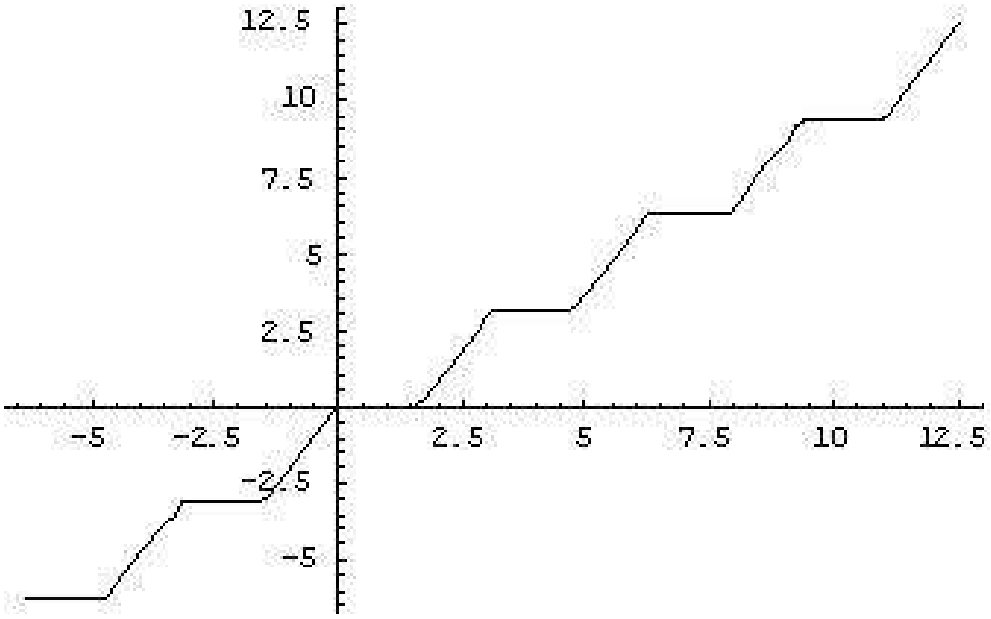
## §5. Smarandache stepped functions

Combining the eccentric ramp functions, of numerical eccentricity $s = 1$, with eccentrically rectangular functions will result the stepped functions, called Smarandache stepped functions, in honor of the Romanian mathematician Florentin Smarandache. Some of these functions, along with their relations of definition, are presented in the following graphs.

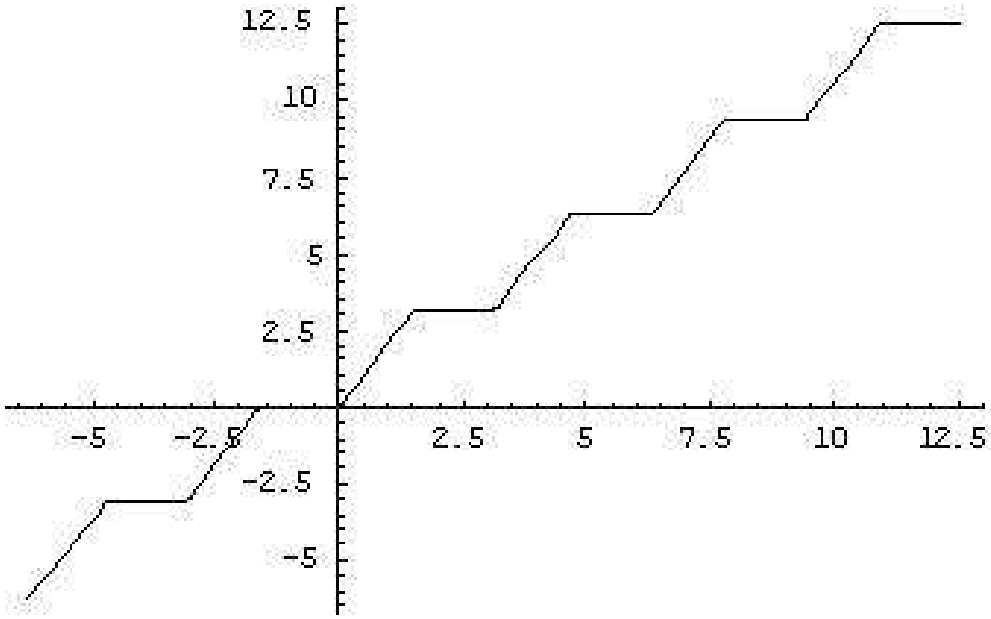
$$Parametric\ [Plot[\{t, t - ArcSin[Sin[t]]\ Cos[t]/Sqrt[1 - Sin[t] \wedge 2])\}, \{t, -2Pi,\ 4Pi\}]$$



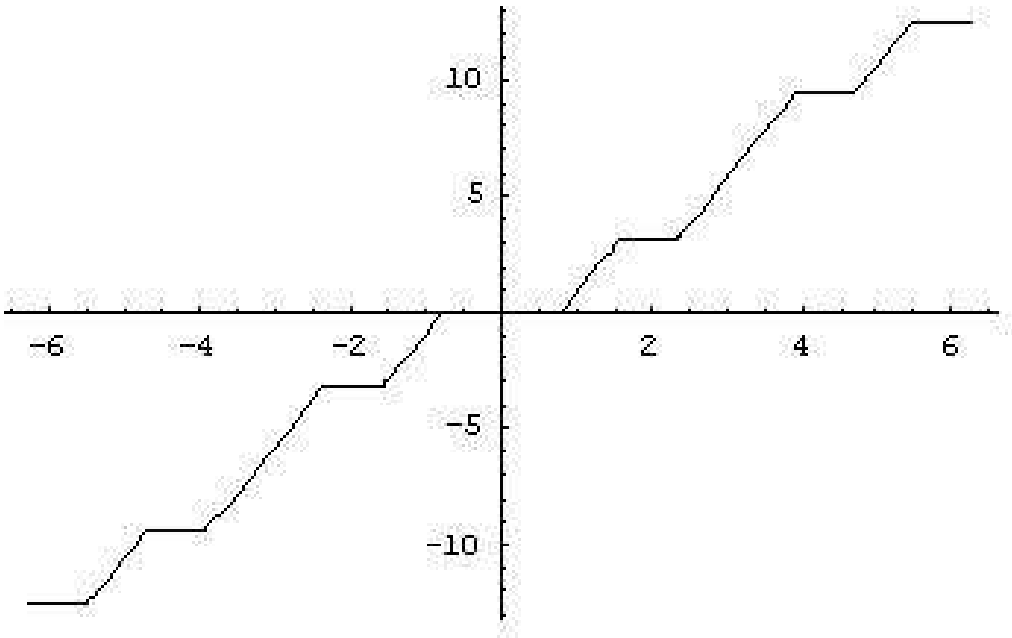

$Parametric[Plot[\{t, t - ArcSin[Sin[t]]\ Sin[t]/qrt[1 - Cos[t] \wedge 2])\}, \{t, -2Pi,\ 4Pi\}]$



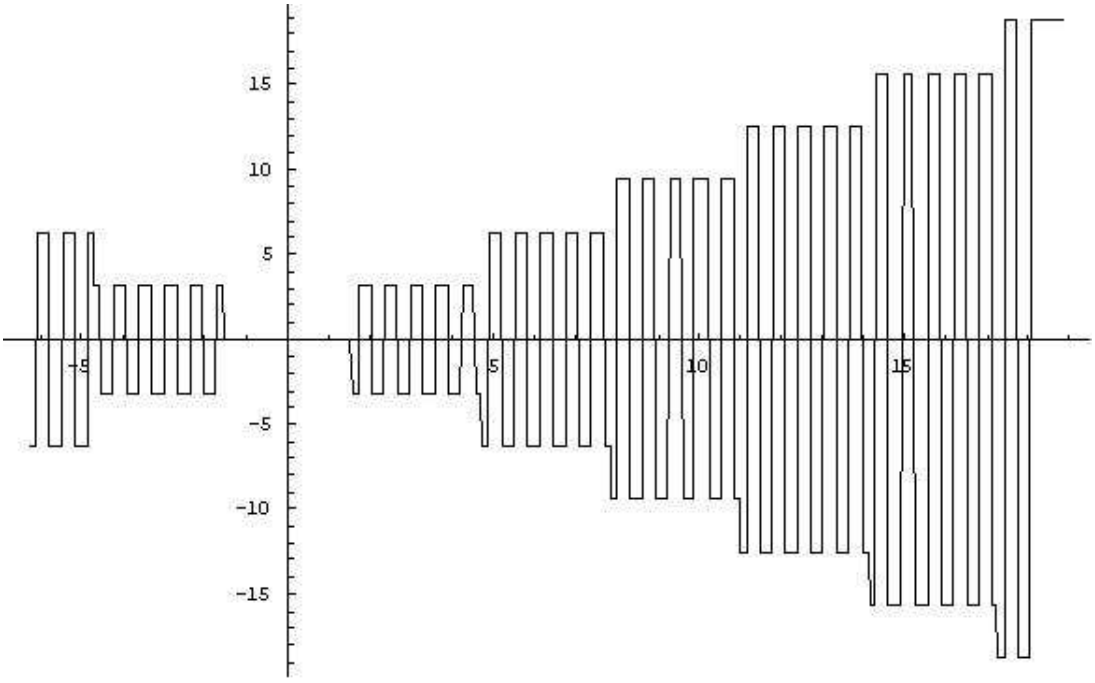

$Parametric[Plot[\{t, t + \ ArcSin[Sin[t]]\ (Sin[t]/Sqrt[1 - Cos[t] \wedge 2])\}, \{t, -2Pi,\ 4Pi\}]$



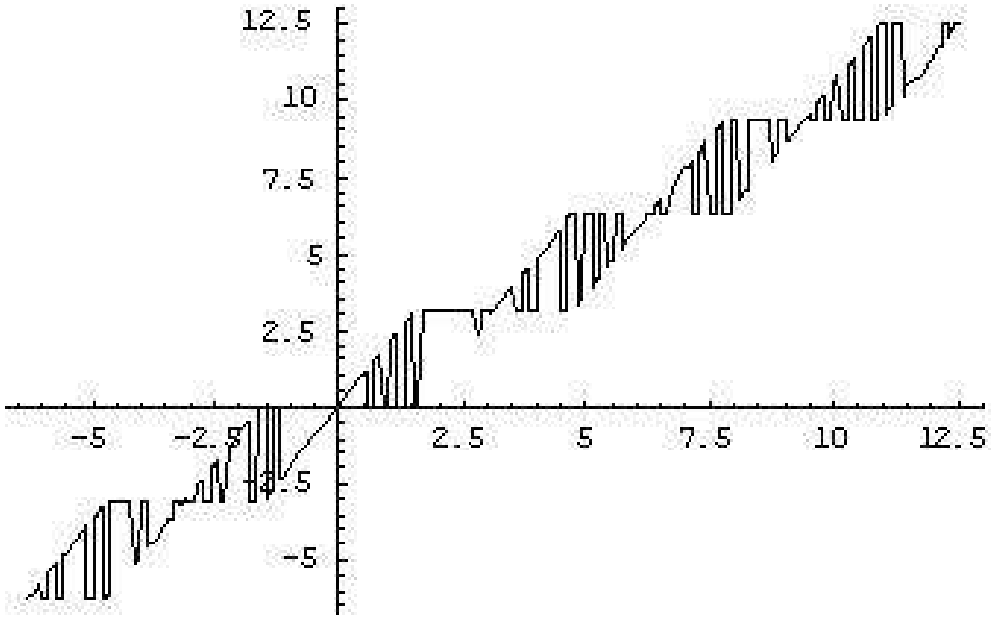

$Parametric[Plot[\{t, 2t - ArcSin[Sin[2t]]\ Cos[t]/Sqrt[1 - Sin[t] \wedge 2])\},\ \{t, -2Pi,\ 2Pi\}]$



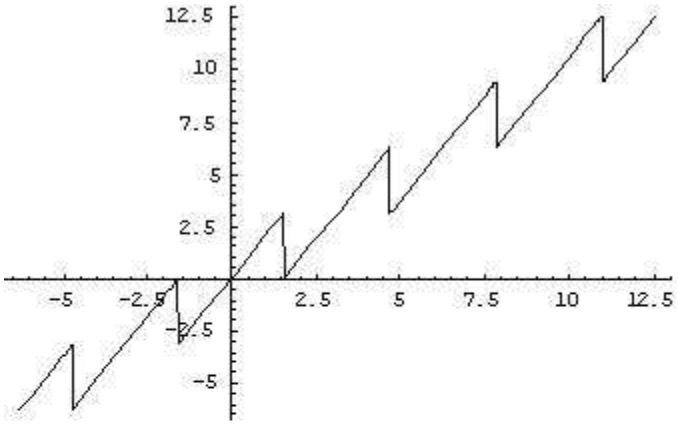

$Parametric[Plot[\{t, (t - ArcSin[Sin[t]]\ (Cos[t]/Sqrt[1 - Sin[t] \wedge 2]))\ (Cos[10t]/Sqrt[1 - Sin[10t] \wedge 2])\},\ \{t, -2Pi,\ 6Pi\}]$



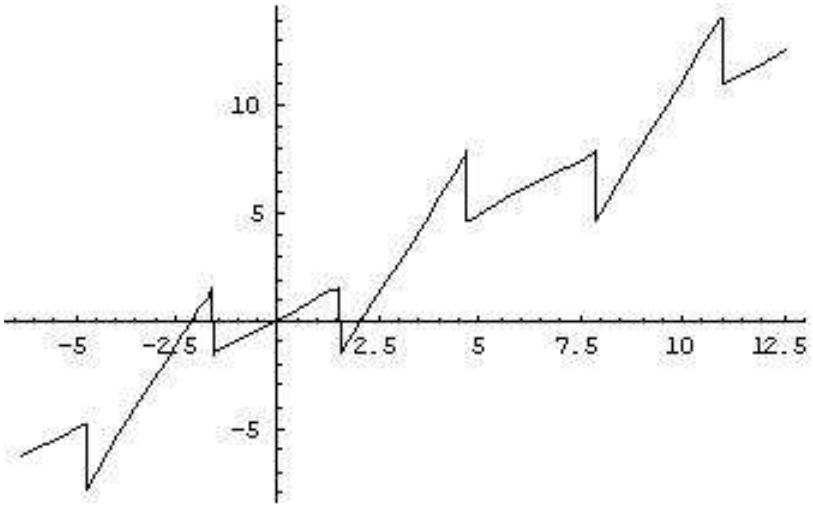

$Parametric[Plot[\{t, t - ArcSin[Sin[t]]\ (Cos[20t]/Sqrt[1 - Sin[20t] \wedge 2])\}, \{t, -2Pi,\ 4Pi\}]$




$Parametric[Plot[\{t, t + ArcSin[Sin[t]]\ (Cos[t]/Sqrt[1 - Sin[t] \wedge 2])\}, \{t, -2Pi,\ 4Pi\}]$




$Parametric[Plot[\{t, t - ArcSin[Sin[t]]\ (1 - Cos[t]/Sqrt[1 - Sin[t] \wedge 2])\}, \{t, -2Pi,\ 4Pi\}]$




$Parametric[Plot[\{t, t - ArcSin[Sin[t]]\ Cos[t]/Sqrt[1 - Sin[t] \wedge 2])) \ (t - ArcSin[Sin[10t]]\ Cos[10t]/Sqrt[1 - Sin10t] \wedge 2])\},\ \{t, -2Pi,\ 4Pi\}]$

## References

[1] Smarandache, Florentin, A Function in Number Theory, <Analele Univ., Timisoara>, University of Timisoara, fascicle 1, **XVIII**(1980), 79-88.

[2] Smarandache, Florentin, Sequences of Numbers Involved in Unsolved Problems, Hexis, 2006, 139.

[3] Smarandache, Florentin, An Infinity of Unsolved Problems Concerning a Function in the Number Theory, <The book of the abstracts of the short communications>, Section 03 (Number Theory), International Congress of Mathematicians, University of Berkeley, CA, USA,

1986, 77.

[4] Smarandache, Florentin, A Numerical Function in the Congruence Theory, <Libertas Mathematica>, University of Texas at Arlington, **XII**(1992), 181-5.

[5] Smarandache, Florentin, Definitions, solved and unsolved problems, conjectures, and theorems in number theory and elementary geometry, Xiquan Publ. Hse., Phoenix, 2000, 84.

[6] Kecs, W. Teodorescu, P.P., Applicaii ale teoriei distribuiilor În mecanic? The Romanian Academy printing house, Bucharest, 1970.

[7] Schoenberg, J. Isaac, Priveliti matematice - Technical printing house, Bucharest, 1989.

[8] Selariu, Mircea Eugen, Funcii circulare excentrice, Presentation at the $I^{st}$ National Conference of Vibrations in the machines construction, Timisoara, 1978, 101-108.

[9] Selariu, Mircea Eugen, Functii circulare excentrice i extensia lor, Scientific & Technical Bulletin of the Polytechnic Institute TV, Timisoara, Series Mec., Tom. 25(39), Fasc. 1, 1981, 189-196.

[10] Selariu, Mircea Eugen, Supermatematica- Communication at the VII International Conference of Eng., Mech. and Tech. TEHNO'95, Timisoara, Math. Appl., **9**, 41-64.

[11] Selariu, Mircea Eugen, Funcii supermatematice circulare excentrice de variabil? centric?, The VIII International Conference of Eng., Mech. and Tech. TEHNO'98, Timisoara, pp. 531-548.

[12] Selariu, Mircea Eugen, Quadrilobic Vibration Systems, The$11^{th}$ International Conference on Vibration Engineering, Timisoara, Romania, Sep. 27-30, 2005.

[13] Selariu, Mircea Eugen, Introducerea strămbei În matematic?, The National Symposium of Gh. Anghel University, Drobeta Turnu Severin, May 2003, 171-178.

[14] Savescu, M. Constantin, I. Petrescu, T., Metode de aproximare În analiza circuitelor electronice, Technical printing house, Bucharest, 1982.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 93-97**

# Sequence of numbers with alternate common differences

Xiong Zhang and Yilin Zhang

Shaanxi Institute of Education
Xi'an, Shaanxi, P.R.China

Received Oct. 28, 2006

**Abstract** In this paper, two types of special sequences of numbers are discussed. One is the number sequence with alternate common differences; and the other, as a generalization of the first one, is the periodic sequences of numbers with two common differences. The formulas of the general term $a_n$ and the sum of the first $n$ terms $S_n$ are given respectively.

**Keywords** Sequence of numbers with alternate common differences, periodic number sequence with two common differences, general term $a_n$, the sum of the first $n$ terms $S_n$.

## §1. Sequence of numbers with alternate common differences

**Definition 1.1.** A sequence of numbers $\{a_n\}$ is called a sequence of numbers with alternate common differences if the following conditions are satisfied:

$(i)$ $\forall k \in N,\ \ a_{2k} - a_{2k-1} = d_1$,

$(ii)$ $\forall k \in N,\ \ a_{2k+1} - a_{2k} = d_2$,

here $d_1$ ($d_2$) is called the first (the second) common differences of $\{a_n\}$.

**Example 1.2.** The number sequence 1, 3, 4, 6, 7, 9, 10, 12, 13, $\cdots$, is a sequence of numbers with alternate common differences, where $d_1 = 2$, $d_2 = 1$.

In this section, $\{a_n\}$ denotes a sequence of numbers with alternate common differences $d_1$ and $d_2$. $S_n$ denotes the sum of the first $n$ terms of $\{a_n\}$.

Obviously, $\{a_n\}$ has the following form:

$$a_1, a_1 + d_1, a_1 + d_1 + d_2, a_1 + 2d_1 + d_2, a_1 + 2d_1 + 2d_2, a_1 + 3d_1 + 2d_2, \cdots . \tag{1}$$

**Theorem 1.3.** The formula of general term of $\{a_n\}$ is

$$a_n = a_1 + \left[\frac{n}{2}\right] d_1 + \left[\frac{n-1}{2}\right] d_2, \tag{2}$$

where $[x]$ means the biggest integer less than or equal to $x$.

**Proof.** We prove this theorem by using induction on $n$.

Obviously, (2) holds when $n = 1,\ 2$.

Suppose (2) holds when $n=k$, $a_k = a_1 + \left[\frac{k}{2}\right] d_1 + \left[\frac{k-1}{2}\right] d_2$.

(i) If $k = 2m-1$, then

$$\begin{aligned} a_{k+1} &= a_k + d_1 \\ &= a_1 + \left[\frac{k}{2}\right] d_1 + \left[\frac{k-1}{2}\right] d_2 + d_1 \\ &= a_1 + (m-1)d_1 + (m-1)d_2 + d_1 \\ &= a_1 + \left[\frac{2m}{2}\right] d_1 + \left[m - 1 + \frac{1}{2}\right] d_2 \\ &= a_1 + \left[\frac{k+1}{2}\right] d_1 + \left[\frac{(k+1)-1}{2}\right] d_2; \end{aligned}$$

(ii) If $k = 2m$, then

$$\begin{aligned} a_{k+1} &= a_k + d_2 \\ &= a_1 + \left[\frac{k}{2}\right] d_1 + \left[\frac{k-1}{2}\right] d_2 + d_2 \\ &= a_1 + md_1 + (m-1)d_2 + d_2 \\ &= a_1 + md_1 + md_2 \\ &= a_1 + \left[\frac{k+1}{2}\right] d_1 + \left[\frac{(k+1)-1}{2}\right] d_2. \end{aligned}$$

Therefore, (2) holds when $n = k+1$. This proves Theorem 1.3.

**Theorem 1.4.** The formula of the general term of $\{a_n\}$ can also be

$$a_n = a_1 + \frac{(n-1)(d_1+d_2)}{2} + \frac{[1+(-1)^n](d_1-d_2)}{2}.$$

**Theorem 1.5.**

$$a_n = na_1 + \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] d_1 + \left[\frac{n-1}{2}\right]\left[\frac{n}{2}\right] d_2.$$

**Proof.**

$$\begin{aligned} S_n &= a_1 + (a_1+d_1) + (a_1+d_1+d_2) + (a_1+2d_1+d_2) + (a_1+2d_1+2d_2) + \cdots \\ &= a_1 + (1-1)(d_1+d_2) + (a_1+d_1) + (1-1)(d_1+d_2) + a_1 + (2-1)(d_1+d_2) \\ &\quad + (a_1+d_1) + (2-1)(d_1+d_2) + a_1 + (3-1)(d_1+d_2) + (a_1+d_1) + (3-1) \\ &\quad (d_1+d_2) + \cdots + a_1 + \left(\left[\frac{n+1}{2}\right] - 1\right)(d_1+d_2) + (a_1+d_1) + \left(\left[\frac{n}{2}\right] - 1\right)(d_1+d_2) \\ &= \frac{1}{2}\left[\frac{n+1}{2}\right]\left(a_1 + a_1 + \left(\left[\frac{n+1}{2}\right] - 1\right)(d_1+d_2)\right) + \frac{1}{2}\left[\frac{n}{2}\right]((a_1+d_1) + a_1 + d_1 \\ &\quad + \left(\left[\frac{n}{2}\right] - 1\right)(d_1+d_2)) \\ &= \frac{1}{2}\left[\frac{n+1}{2}\right]\left(2a_1 + \left[\frac{n-1}{2}\right](d_1+d_2)\right) + \frac{1}{2}\left[\frac{n}{2}\right]\left(2a_1 + 2d_1 + \left[\frac{n-2}{2}\right](d_1+d_2)\right) \\ &= \left(\left[\frac{n}{2}\right] + \left[\frac{n+1}{2}\right]\right)a_1 + \frac{1}{2}\left(\left[\frac{n+1}{2}\right]\left[\frac{n-1}{2}\right] + \left[\frac{n}{2}\right]\left[\frac{n-2}{2}\right] + 2\left[\frac{n}{2}\right]\right)d_1 \end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}\left(\left[\frac{n+1}{2}\right]\left[\frac{n-1}{2}\right]+\left[\frac{n}{2}\right]\left[\frac{n-2}{2}\right]\right)d_2 \\
= \; & na_1+\frac{1}{2}\left(\left[\frac{n+2}{2}\right]\left[\frac{n}{2}\right]+\left[\frac{n+1}{2}\right]\left[\frac{n-1}{2}\right]\right)d_1+ \\
& \frac{1}{2}\left(\left[\frac{n+1}{2}\right]\left[\frac{n-1}{2}\right]+\left[\frac{n}{2}\right]\left[\frac{n-2}{2}\right]\right)d_2 \\
= \; & na_1+\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]d_1+\left[\frac{n-1}{2}\right]\left[\frac{n}{2}\right]d_2.
\end{aligned}$$

## §2. Sequence of numbers with alternate common differences

**Definition 2.1.** A sequence of numbers $\{a_n\}$ is called a periodic number sequence with two common differences if the following conditions are satisfied:

$(i)$ $\forall k=0,\ 1,\ 2,\ \cdots,\ a_{kt+1},\ a_{kt+2},\ a_{kt+3},\ \cdots,\ a_{kt+t}$ is a finite arithmetic progression with $d_1$ as the common difference, where $t$ is a constant natural numbers;

$(ii)$ $\forall k=0,\ 1,\ 2,\ \cdots,\ a_{(k+1)t+1}=a_{(k+1)t}+d_2$.

We call the finite arithmetic progression $a_{kt+1},\ a_{kt+2},\ a_{kt+3},\ \cdots,\ a_{kt+t}$ is the $(k+1)$th periodic of $\{a_n\}$ and $a_{(k+1)t},\ a_{(k+1)t+1}$ is the $(k+1)$th interval of $\{a_n\}$; $d_1$ is named the common difference inside the periods and $d_2$ is called the interval common difference, $t$ is called the number sequence $\{a_n\}$'s period.

In this section, $\{a_n\}$ denotes a periodic sequence of numbers with two common differences $d_1$ and $d_2$. It's easy to get that $\{a_n\}$ has the following form:

$$\begin{aligned}
& a_1,\ a_1+d_1,\ a_1+2d_1,\ \cdots,\ a_1+(t-1)d_1;\ a_1+(t-1)d_1+d_2,\ a_1+td_1+d_2, \\
& a_1+(t+1)d_1+d_2,\ \cdots,\ a_1+(2t-2)d_1+d_2;\ a_1+(2t-2)d_1+2d_2, \\
& a_1+(2t-1)d_1+2d_2,\ a_1+2td_1+2d_2,\ \cdots,\ a_1+(3t-3)d_1+d_2;\ \cdots. \qquad (3)
\end{aligned}$$

Particularly, when $t=2$, $\{a_n\}$ becomes a sequences of numbers with alternate common differences $d_1$ and $d_2$. So the concept of a periodic sequence of numbers with two common differences is a generalization of the concept of a numbers sequence with alternate common differences.

**Theorem 2.2.** The formula of general term of (3) is

$$a_n=a_1+\left(n-1-\left[\frac{n-1}{t}\right]\right)d_1+\left[\frac{n-1}{t}\right]d_2.$$

**Proof.**

$$\begin{aligned}
a_n &= a_1+(n-1)d_1+(d_2-d_1)k \\
&= a_1+kd_2+[(n-1)-k]d_1 \\
&= a_1+\left(n-1-\left[\frac{n-1}{t}\right]\right)d_1+\left[\frac{n-1}{t}\right]d_2,
\end{aligned}$$

here $k$ means the number of intervals, it can be proved easily that $k=\left[\dfrac{n-1}{t}\right]$.

**Theorem 2.3.** $\{a_n\}$ is a periodic sequence of numbers with two common differences $d_1$ and $d_2$, the sum of the first $n$ terms of $\{a_n\}$ $S_n$ is

$$\begin{aligned} S_n &= na_1 + \frac{t(t-1)}{2}\left[\frac{n}{t}\right]^2 d_1 + \frac{\left[\frac{n}{t}\right]\left(\left[\frac{n}{t}\right]-1\right)}{2}td_2 + \left(\left[\frac{n}{t}\right](t-1)d_1 + \left[\frac{n}{t}\right]d_2\right)\cdot \\ &\quad \left(n - \left[\frac{n}{t}\right]t\right) + \frac{\left(n-\left[\frac{n}{t}\right]t\right)\left(n-\left[\frac{n}{t}\right]t-1\right)}{2}d_1. \end{aligned}$$

Particularly, when $t|2$, suppose $\dfrac{n}{t} = k$, then

$$S_n = na_1 + \frac{t(t-1)}{2}k^2 d_1 + \frac{k(k-1)}{2}td_2$$

**Proof.** Let $M_{(k,t)}$ be the sum of the $t$ terms of $(k+1)$th period. Then

$$\begin{aligned} M_{(k,t)} &= ta_{(k+1)t+1} + \frac{t(t-1)}{2}d_1 \\ &= t[a_1 + (k-1)d_2 + ((k-1)t-(k-1))d_1] + \frac{t(t-1)}{2}d_1 \\ &= ta_1 + t(k-1)d_2 + \frac{2k-1}{2}t(t-1)d_1, \end{aligned}$$

$$\begin{aligned} M_{(k,t)} &= ta_{kt+1} + \frac{t(t-1)}{2}d_1 \\ &= t(a_1 + kd_2 + (kt-k)d_1) \\ &= ta_1 + tkd_2 + \frac{2k+1}{2}t(t-1)d_1. \end{aligned}$$

Hence

$$M_{(k+1,t)} - M_{(k,t)} = t(t-1)d_1 + td_2.$$

Therefore the new sequence $\{M_{(k,t)}\}$ generated from $\{a_n\}$ is an arithmetic progression with

$$M_{(k,t)} = ta_1 + \frac{t(t-1)}{2}d_1,$$

$$d = t(t-1)d_1 + td_2.$$

So the sum of the first $\left[\dfrac{n}{t}\right]t$ terms of $\{a_n\}$.

$$\begin{aligned} S_{\left[\frac{n}{t}\right]t} &= \left(ta_1 + \frac{t(t-1)}{2}d_1\right)\left[\frac{n}{t}\right] + \frac{\left[\frac{n}{t}\right]\left(\left[\frac{n}{t}\right]-1\right)}{2}(t(t-1)d_1 + td_2) \\ &= \left[\frac{n}{t}\right]ta_1 + \frac{t(t-1)}{2}\left[\frac{n}{t}\right]^2 d_1 + \frac{\left[\frac{n}{t}\right]\left(\left[\frac{n}{t}\right]-1\right)}{2}td_2. \end{aligned}$$

and

$$\begin{aligned} S_n - S_{\left[\frac{n}{t}\right]t} &= a_{\left[\frac{n}{t}\right]t+1}\left(n - \left[\frac{n}{t}\right]t\right) + \frac{\left(n-\left[\frac{n}{t}\right]t\right)\left(n-\left[\frac{n}{t}\right]t-1\right)}{2}d_1 \\ &= \left(a_1 + \left[\frac{n}{t}\right](t-1)d_1 + \left[\frac{n}{t}\right]d_2\right)\left(n - \left[\frac{n}{t}\right]t\right) + \frac{\left(n-\left[\frac{n}{t}\right]t\right)\left(n-\left[\frac{n}{t}\right]-1\right)}{2}d_1, \end{aligned}$$

thus,

$$\begin{aligned} S_n &= S_{\left[\frac{n}{t}\right]t} + (S_n - S_{\left[\frac{n}{t}\right]t}) \\ &= na_1 + \frac{t(t-1)}{2}\left[\frac{n}{t}\right]^2 d_1 + \frac{\left[\frac{n}{t}\right]\left(\left[\frac{n}{t}\right]-1\right)}{2}td_2 + \left(\left[\frac{n}{t}\right](t-1)d_1 + \left[\frac{n}{t}\right]d_2\right)\cdot \\ &\quad \left(n - \left[\frac{n}{t}\right]t\right) + \frac{\left(n-\left[\frac{n}{t}\right]t\right)\left(n-\left[\frac{n}{t}\right]t-1\right)}{2}d_1. \end{aligned}$$

Particularly, when $t|n$, suppose $\frac{n}{t} = k$, then

$$S_n = na_1 + \frac{t(t-1)}{2}k^2 d_1 + \frac{k(k-1)}{2}td_2.$$

# References

[1] Pan Chengdong and Pan Chengbiao, Foundation of Analytic Number Theory, Science Press, Beijing, 1999, 202-205.

[2] Pan Chengdong and Pan Chengbiao, Elementary Number Theory, Beijing University Press, Beijing, 1992.

[3] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 98-101**

# On the near pseudo Smarandache function

Yongfeng Zhang

Department of Mathematics, Xianyang Normal College,
Xianyang, Shaanxi, P.R.China

Received June 13, 2006

**Abstract** For any positive integer $n$, the near pseudo Smarandache function $K(n)$ is defined as $K(n) = m = \frac{n(n+1)}{2} + k$, where $k$ is the smallest positive integer such that $n$ divides $m$. The main purpose of this paper is using the elementary method to study the calculating problem of an infinite series involving the near pseudo Smarandache function $K(n)$, and give an exact calculating formula.

**Keywords** Near pseudo Smarandache function, infinite series, exact calculating formula.

## §1. Introduction and results

For any positive integer $n$, the near pseudo Smarandache function $K(n)$ is defined as follows:

$$K(n) = m,$$

where $m = \frac{n(n+1)}{2} + k$, and $k$ is the smallest positive integer such that $n$ divides $m$.

The first few values of $K(n)$ are $K(1) = 2$, $K(2) = 4$, $K(3) = 9$, $K(4) = 12$, $K(5) = 20$, $K(6) = 24$, $K(7) = 35$, $K(8) = 40$, $K(9) = 54$, $K(10) = 50$, $K(11) = 77$, $K(12) = 84$, $K(13) = 104$, $K(14) = 112$, $K(15) = 135$, $\cdots$. This function was introduced by A.W.Vyawahare and K.M.Purohit in [1], where they studied the elementary properties of $K(n)$, and obtained a series interesting results. For example, they proved that 2 and 3 are the only solutions of $K(n) = n^2$; If $a$, $b > 5$, then $K(a \cdot b) > K(a) \cdot K(b)$; If $a > 5$, then for all positive integer $n$, $K(a^n) > n \cdot K(a)$; The Fibonacci numbers and the Lucas numbers do not exist in the sequence $\{K(n)\}$; Let $C$ be the continued fraction of the sequence $\{K(n)\}$, then $C$ is convergent and $2 < C < 3$; $K(2^n - 1) + 1$ is a triangular number; The series $\sum_{n=1}^{\infty} \frac{1}{K(n)}$ is convergent. The other contents related to the near pseudo Smarandache function can also be found in references [2], [3] and [4].

In this paper, we use the elementary method to study the calculating problem of the series

$$\sum_{n=1}^{\infty} \frac{1}{K^s(n)}, \tag{1}$$

and give an exact calculating formula for (1). That is, we shall prove the following conclusion:

**Theorem.** For any real number $s > \frac{1}{2}$, the series (1) is convergent, and

(a) $$\sum_{n=1}^{\infty}\frac{1}{K(n)}=\frac{2}{3}\ln 2+\frac{5}{6};$$

(b) $$\sum_{n=1}^{\infty}\frac{1}{K^2(n)}=\frac{11}{108}\cdot\pi^2-\frac{22+2\ln 2}{27}.$$

In fact for any positive integer $s$, using our method we can give an exact calculating formula for (1), but the calculation is very complicate if $s$ is large enough.

## §2. Proof of the theorem

In this section, we shall prove our theorem directly. In fact for any positive integer $n$, it is easily to deduce that $K(n)=\frac{n(n+3)}{2}$ if $n$ is odd and $K(n)=\frac{n(n+2)}{2}$ if $n$ is even. So from this properties we may immediately get

$$\frac{n^2}{2}<K(n)<\frac{(n+3)^2}{2},$$

or

$$\frac{1}{(n+3)^{2s}}\ll\frac{1}{K^s(n)}\ll\frac{1}{n^{2s}}.$$

So the series (1) is convergent if $s>\frac{1}{2}$.

Now from the properties of $K(n)$ we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{K(n)} &= \sum_{n=1}^{\infty}\frac{1}{K(2n-1)}+\sum_{n=1}^{\infty}\frac{1}{K(2n)}\\ &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)(n+1)}+\sum_{n=1}^{\infty}\frac{1}{2n(n+1)}\\ &= \frac{2}{3}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n+2}\right)+\frac{1}{2}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \frac{2}{3}\cdot\lim_{N\to\infty}\left(\sum_{n\le N}\frac{1}{2n-1}-\sum_{n\le N}\frac{1}{2n+2}\right)+\frac{1}{2}\\ &= \frac{2}{3}\cdot\lim_{N\to\infty}\left(\sum_{n\le 2N}\frac{1}{n}-\frac{1}{2N+2}+\frac{1}{2}-\sum_{n\le N}\frac{1}{n}\right)+\frac{1}{2}. \qquad (2)\end{aligned}$$

Note that for any $N>1$, we have the asymptotic formula (See Theorem 3.2 of [5])

$$\sum_{n\le N}\frac{1}{n}=\ln N+\gamma+O\left(\frac{1}{N}\right), \qquad (3)$$

where $\gamma$ is the Euler constant.

Combining (2) and (3) we may immediately obtain

$$\sum_{n=1}^{\infty}\frac{1}{K(n)}=\frac{2}{3}\cdot\lim_{N\to\infty}\left[\ln(2N)+\gamma+\frac{1}{2}-\ln N-\gamma+O\left(\frac{1}{N}\right)\right]+\frac{1}{2}=\frac{2}{3}\ln 2+\frac{5}{6}.$$

This completes the proof of (a) in Theorem.

Now we prove (b) in Theorem. From the definition and properties of $K(n)$ we also have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{K^2(n)} &= \sum_{n=1}^{\infty}\frac{1}{K^2(2n-1)}+\sum_{n=1}^{\infty}\frac{1}{K^2(2n)}\\ &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2(n+1)^2}+\sum_{n=1}^{\infty}\frac{1}{(2n)^2(n+1)^2}.\end{aligned}\tag{4}$$

Note that the identities

$$\frac{1}{(2n-1)^2(n+1)^2}=\frac{2}{27}\left(\frac{1}{2n+2}-\frac{1}{2n-1}\right)+\frac{1}{9}\frac{1}{(2n-1)^2}+\frac{1}{9}\frac{1}{(2n+2)^2},\tag{5}$$

$$\frac{1}{n^2(n+1)^2}=2\left(\frac{1}{n+1}-\frac{1}{n}\right)+\frac{1}{n^2}+\frac{1}{(n+1)^2},\tag{6}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}\quad\texttt{and}\quad\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}=\frac{\pi^2}{6}-1.\tag{7}$$

From (3), (4), (5), (6) and (7) we may deduce that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{K^2(n)} &= \frac{2}{27}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{2n+2}-\frac{1}{2n-1}\right)+\frac{1}{9}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)^2}+\frac{1}{(2n+2)^2}\right)\\ &\quad+\frac{1}{2}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n}\right)+\frac{1}{4}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}\right)\\ &= \frac{2}{27}\cdot\lim_{N\to\infty}\left[\sum_{n\le N}\frac{1}{2n+2}-\sum_{n\le N}\frac{1}{2n-1}\right]+\frac{\pi^2}{72}+\frac{\pi^2}{216}-\frac{1}{36}\\ &\quad+\frac{1}{2}\cdot\lim_{N\to\infty}\left[\sum_{n\le N}\frac{1}{n+1}-\sum_{n\le N}\frac{1}{n}\right]+\frac{\pi^2}{24}+\frac{\pi^2}{24}-\frac{1}{4}\\ &= \frac{2}{27}\cdot\lim_{N\to\infty}\left[-\frac{1}{2}+\ln N-\ln(2N)+O\left(\frac{1}{N}\right)\right]+\frac{\pi^2}{54}-\frac{1}{36}\\ &\quad-\frac{1}{2}+\frac{\pi^2}{12}-\frac{1}{4}\\ &= \frac{11}{108}\cdot\pi^2-\frac{22+2\ln 2}{27}.\end{aligned}$$

This completes the proof of (b) in Theorem.

# References

[1] A.W.Vyawahare and K.M.Purohit, Near Pseudo Smarandache Function, Smarandache Notions Journal, **14**(2004), 42-59.

[2] C.Ashbacher, Introduction to Smarandache functions, Journal of Recreational Mathematics, 1996, 249.

[3] David Gorski, The Pseudo Smarandache Functions, Smarandache Notions Journal, **12**(2000), 104.

[4] Castrillo Jose, Smarandache Continued Fractions, Smarandache Notions Journal, **9**(1998), 40.

[5] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

**Scientia Magna**
**Vol. 3 (2007), No. 1, 102-107**

# Smarandache mukti-squares[1]

Arun S. Muktibodh
Mohota Science College, Umred Rd., Nagpur-440009, India.
Email: amukti2000@yahoo.com

Received Feb 1, 2007.

**Abstract** In [4] we have introduced Smarandache quasigroups which are Smarandache non-associative structures. A quasigroup is a groupoid whose composition table is a Latin square. There are squares in the Latin squares which seem to be of importance to study the structure of Latin Squares. We consider the particular type of squares properly contained in the Latin squares which themselves contain a Latin square. Such Latin squares are termed as Smarandache Mukti-Squares or SMS. Extension of some SMS to Latin squares is also considered.

## §1. Introduction

Latin squares were first studied by Euler near the end of eighteenth century. A Latin square of order $n$ is an $n$ by $n$ array containing symbols from some alphabet of size $n$, arranged so that each symbol appears exactly once in each row and exactly once in each column. Orthogonal Latin squares play an important role in the development of the theory of Latin squares. The best introduction of Latin Squares is in Bose and Manvel [1]. Today Latin squares have wide applications varying from 'Experimental Designs' in Agriculture to Cryptography and Computer science. There are some typical squares properly contained in some Latin squares. These squares themselves contain a Latin square. We have termed such squares as Smarandache Mukti-Squares. In this paper, we are initiating the study of such squares. We prove some properties and some important results.

**Definition 1.1.** An $n \times n$ array containing symbols from some alphabet of size $m$ with $m \geq n$ is called a square of order $n$.

**Definition 1.2.** A Latin square of order $n$ is an n by n array containing symbols from some alphabet of size $n$, arranged so that each symbol appears exactly once in each row and exactly once in each column.

**Definition 1.3.** If a Latin square $L$ contains a Latin square $S$ properly, then $S$ is called a sub-Latin square.

**Definition 1.4.** A square in which ;

1. No element in the first row is repeated.
2. No element in the first column is repeated.

[1]Acknowledgement: UGC, India has supported this work under project No.23- 245/06.

3. Elements in first row and first column are same, is called a Mukti-Square.

**Example 1.1.** The following are Mukti-Squares of order 3 and 4 with alphabets $\{0,\ 1,\ 2,\ 3,\ 4\}$ and $\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}$ respectively.

$$\begin{matrix} 3 & 1 & 0 \\ 0 & 3 & 4 \\ 1 & 2 & 3 \end{matrix}$$

and

$$\begin{matrix} 4 & 5 & 6 & 7 \\ 5 & 3 & 1 & 2 \\ 6 & 1 & 5 & 3 \\ 7 & 2 & 3 & 1 \end{matrix}$$

**Definition 1.5.** If a square contains a Latin Square properly then the square is called a Smarandache Mukti-Square or SMS.

**Example 1.2.** The following is a Smarandache Mukti-Square of order 4.

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 3 & 1 \\ 4 & 3 & 1 & 2 \end{matrix}$$

Clearly Mukti-Square on $\{1, 2, 3, 4\}$ contains a Latin square

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix}$$

on $\{1, 2, 3\}$.

**Remark.**

1. Any Latin square can be rotated about the axis through its center perpendicular to its plane.

2. The angles of rotation are $0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$.

**Theorem 1.1.** Order of an SMS is greater than or equal to 3.

**Proof.** Follows from the definition of SMS.

## §2. Orthogonal

Smarandache Mukti-squares Two Smarandache Mukti-Squares are said to be orthogonal if the Latin squares contained in them are orthogonal. The following SMSs are orthogonal to

each other.

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 3 & 1 \\ 4 & 3 & 1 & 2 \end{matrix}$$

is orthogonal to

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 4 & 3 & 1 & 2 \\ 3 & 2 & 3 & 1 \end{matrix}$$

## §3. Latin squares which contain SMS

**Theorem 3.1.** If a Latin square has no subLatin square properly contained in it then it has no SMS.

**Proof.** Follows from the definition of SMS.

**Example 3.1.** Consider the Latin squares;

$$\begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{matrix}$$

$$\longrightarrow$$

$$\begin{matrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 3 & 2 \\ 1 & 0 & 2 & 3 \end{matrix}$$

$$\longrightarrow$$

$$\begin{matrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{matrix}$$

$$\longrightarrow$$

$$\begin{array}{cccc} 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{array}$$

Note that each rotation of the Latin square through 0, $\frac{\pi}{2}$, $\pi$, $\frac{3\pi}{2}$ yields a new SMS in the Latin square.

In this case we say that the given Latin square is Fully SMS- Symmetric.

**Theorem 3.2.** If a Latin square of order 4 is fully SMS-symmetric then its ortogonal Latin square can not be fully SMS-symmetric.

**Proof.** Consider a fully SMS-symmetric Latin square of order 4 as:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array}$$

It can be practically verified that the another Latin square which is fully SMS-symmetric is not orthogonal to it.

There do exist Latin squares which contain no SMS. The example follows:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{array}$$

**Theorem 3.3.** A Latin square of order 3 does not posses an SMS.

**Proof.** Follows from the definition of SMS.

**Theorem 3.4.** A Latin square may have more than one SMS. They may be of different orders.

**Proof.** We prove the theorem by giving an example:

$$\begin{array}{ccccccc} 7 & 3 & 2 & 1 & 6 & 5 & 4 \\ 3 & 6 & 1 & 2 & 7 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 & 7 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 4 & 5 & 3 & 1 & 2 \\ 5 & 4 & 7 & 6 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 & 2 & 3 & 1 \end{array}$$

This is a partially SMS-symmetric Latin square. We can see the SMS on {4, 5, 6, 7}. Other SMSs can be seen by rotating the Latin square. There are two SMS on {4, 6, 7} which are identical.

## §4. Extension of Smarandache Mukti-Squares

In this section we have tried to find out which SMSs can be extended to form a Latin square.

**Theorem 4.1.** If a Latin square of order 2 is extended to form a Latin square then the minimum order of such a Latin square will be 4 and it must contain an SMS.

**Proof.** Actual extension of such a Latin square will verify the theorem. Consider an SMS;

$$\begin{matrix} d & a \\ a & d \end{matrix}$$

This can be extended to a Latin square as;

$$\begin{matrix} a & d & c & b \\ d & a & b & c \\ c & b & d & a \\ b & c & a & d \end{matrix}$$

**Theorem 4.2.** An SMS of order three can be extended to a Latin square of order 4 if the Latin square contained in the SMS has one element outside the alphabet of the SMS.

**Proof.** Again we prove the theorem by constructing an example. Consider an SMS;

$$\begin{matrix} a & b & c \\ b & d & a \\ c & a & d \end{matrix}$$

This can be extended to a Latin square as

$$\begin{matrix} a & d & c & b \\ d & a & b & c \\ c & b & d & a \\ b & c & a & d \end{matrix}$$

**Theorem 4.3.** It is not possible to extend an SMS of order 3 to a Latin square if the Latin square contained in the SMS has two elements outside the alphabet of the SMS. It has been practically tried out but could not construct the Latin square.

**Theorem 4.4.** An SMS of order 4 can be extended to a Latin square of order 7 if the Latin square contained in the SMS has order 3 and contains all the elements outside the alphabet of the SMS.

**Proof.** We construct an example. Consider an SMS as

$$\begin{matrix} 4 & 5 & 6 & 7 \\ 5 & 3 & 1 & 2 \\ 6 & 1 & 2 & 3 \\ 7 & 2 & 3 & 1 \end{matrix}$$

Then this SMS can be extended to

$$\begin{array}{ccccccc} 7 & 3 & 2 & 1 & 6 & 5 & 4 \\ 3 & 6 & 1 & 2 & 7 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 & 7 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 4 & 5 & 3 & 1 & 2 \\ 5 & 4 & 7 & 6 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 & 2 & 3 & 1 \end{array}$$

We report some of the observations in this study.

1. Every SMS can not be extended to form a Latin square.
2. Order of the SMS does not divide the order of the Latin square containing it.
3. Number of SMS do not change if the Latin square is rotated through 0, $\frac{\pi}{2}$, $\pi$, $\frac{3\pi}{2}$.

## §5. Some open problems

1. Can we extend an SMS of order 3 when all the elements of the Latin square contained in the SMS are outside the alphabet of the SMS?

2. Can we extend an SMS of order 4 when one element of the Latin square contained in the SMS is outside the alphabet of the SMS?

3. Can we extend an SMS of order 4 when two element of the Latin square contained in the SMS are outside the alphabet of the SMS?

## References

[1] Bose R. and B. Manvel., Introduction to Combinatorial Theory, 1984, 135-149, John Wiley Sons.

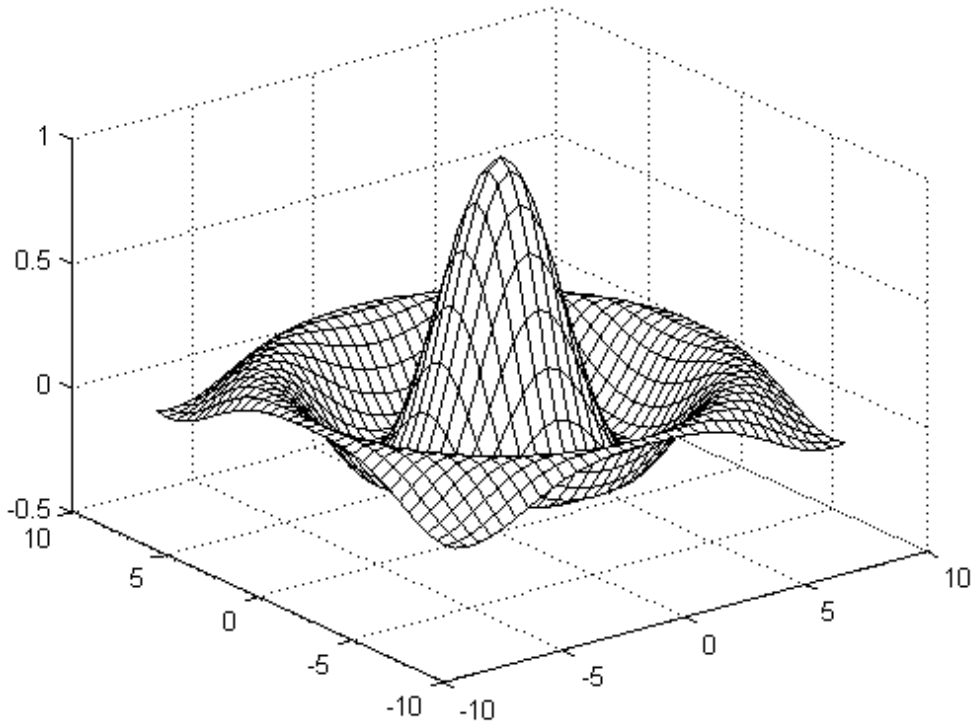
[2] Bose and Bush, Orthogonal Arrays of strength Two and Three, Annals of Statistical Mathematics, **23**(1995), 508-524.

[3] Bush K, Orthogonal Array of Index Unity, Annals of Mathematical Statistics, **23**(1952), 426-434.

[4] Muktibodh A.S., Smarandache Quasigroups, Scientia Magna, **2**(2006), No.1, 13-19.

[5] Shao and Wei, A formula for number of Latin squares, Discrete Mathematics, **110**(1992), 293-296.

[6] Wells B., The number of Latin squares of order 8, Journal of Combinatorial Theory, 1967, 98-99.




# *SCIENTIA MAGNA*

**An international book series**